ON THE BONDAGE NUMBER OF PLANAR AND DIRECTED GRAPHS

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Abstract. The bondage number $b(G)$ of a nonempty graph G is defined to be the cardinality of the smallest set E of edges of G such that the graph $G - E$ has domination number greater than that of G. In this paper we present a simplified proof that $b(G) \leq \min\{8, \Delta(G) + 2\}$ for all planar graphs G, give examples of planar graphs with bondage number 6, and bound the bondage number of directed graphs.

1. Introduction

Given a nonempty graph G, a set D of its vertices is a *dominating set* if every vertex of G is in D or adjacent to a vertex in D. The *dominating number* $\gamma(G)$ of a graph G is defined to be the minimum size of a dominating set of G. We may further define the bondage number of a graph, denoted $b(G)$, to be the cardinality of a smallest set of edges E in G such that $\gamma(G - E) > \gamma(G)$.

The bondage number was first introduced by Bauer et al. [1] in 1983. The two main outstanding conjectures on bondage number were posed by Teschner [8].

Conjecture 1 (Teschner [8]). *If G is a planar graph, then $b(G) \leq \Delta(G) + 1$.*

Conjecture 2 (Teschner [8]). *For any graph G, we have $b(G) \leq \frac{3}{2}\Delta(G)$.*

In 2000, Kang and Yuan [6] showed that $b(G) \leq \min\{8, \Delta(G) + 2\}$ for any planar graph; in Section 2 we will present a simpler proof of this fact.

In [3], Fischermann, Rautenbach, and Volkmann ask whether there exist planar graphs of bondage number 6, 7, or 8. In Section 3, we show that the corona $G = H \circ K_1$ satisfies $b(G) = \delta(H) + 1$, where $\delta(H)$ is the minimum degree in H. In particular, this construction gives us a class of planar graphs with bondage number 6.

In [2], it was originally conjectured that Conjecture 1 held for any graph G; however, Teschner disproved this claim in [7], and Hartnell and Rall [5] showed that for the cartesian product $G_n = K_n \times K_n$, $n > 1$, we have $b(G_n) = \frac{3}{2}\Delta(G_n)$. This led to the formulation of Conjecture 2. Teschner [8] proved that Conjecture 2 holds when $\gamma(G) \leq 3$. In Section 4, we define the bondage number for directed graphs and prove that the directed graph analogue to Conjecture 2 holds.

Throughout the paper, all graphs will be considered finite and nonempty. Furthermore, all undirected graphs will be simple. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively. We denote the degree of a vertex $u \in G$ by $d(u)$, the maximum degree of any vertex in G by $\Delta(G)$, and the minimum degree by $\delta(G)$. The distance between two vertices u and v is denoted $d(u, v)$.

2. A simple proof that $b(G) \leq \min\{8, \Delta(G) + 2\}$ for G planar

In this section we present simple proofs of two theorems originally proved by Kang and Yuan [6]. These proofs rely on a simple application of Euler's formula, and are much shorter than the originals.

We will use the following simple lemmas to bound the bondage numbers of planar graphs.

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Lemma 3 (Hartnell and Rall [5]). *If G is a graph, then for every pair of adjacent vertices u and v in G, $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$. In particular, this implies $b(G) \leq \delta(G) + \Delta(G) - 1$.*

Lemma 4 (Euler's Formula). *Suppose that G is a connected graph which can be embedded on the oriented surface of genus g. Then $|V(G)| - |E(G)| + |F(G)| = 2 - 2g$, where $F(G)$ is the face set of any embedding of G on the surface of genus g.*

We specifically note that planar graphs are those which can be embedded on the sphere, the oriented surface of genus 0. Thus, for planar graphs, we have $|V(G)| - |E(G)| + |F(G)| = 2$.

Theorem 5 (Kang and Yuan [6]). *Let G be a connected planar graph. Then $b(G) \leq \Delta(G) + 2$.*

Proof. Suppose that G is a planar graph. By Lemma 3, if G has any vertices of degree 3 or less, we have $\delta(G) \leq 3$, and Theorem 5 holds. Thus, we can assume $\Delta(G) \geq \delta(G) \geq 4$. For the sake of contradiction, assume $b(G) \geq \Delta(G) + 3$. To each edge $e_i = xy$ in $E(G)$, we assign variables $v_i = \frac{1}{d(x)} + \frac{1}{d(y)}$ and $f_i = \frac{1}{a_1} + \frac{1}{a_2}$, where a_1 and a_2 are the numbers of edges comprising the faces which e_i borders. It is clear that $\sum v_i = |V(G)|$ and $\sum f_i = |F(G)|$, so we have $\sum(v_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 2$, by Lemma 4.

But now, for each i, consider the quantity $v_i + f_i - 1$. If either $d(x)$ or $d(y)$ is equal to 4, by Lemma 3 the other must be equal to $\Delta \geq 4$ and x and y can have no common neighbors, so that a_1 and a_2 are both at least 4. This yields $v_i + f_i - 1 \leq 0$. Suppose one of $d(x)$ and $d(y)$, without loss of generality $d(x)$, is equal to 5. If $d(y) = \Delta - 1 = 4$, then we are in the previous case. Otherwise, we have $d(y) = \Delta \geq 5$, and at most one of a_1 and a_2 equal to 3, so we again get $v_i + f_i - 1 \leq 0$. The only remaining case is $d(x), d(y) \geq 6$, but as $a_1, a_2 \geq 3$, in any case we again obtain $v_i + f_i - 1 \leq 0$. But then summing over all i yields $\sum(v_i + f_i - 1) \leq 0$, which contradicts Euler's formula. □

Considering planar graphs as those which can be embedded on a sphere, it is natural to consider the generalization to graphs which we can embed on surfaces of higher genus. This works for graphs we can embed on the torus, for which $|V(G)| - |E(G)| + |F(G)| = 0$. However, the proof method does not generalize to all graphs, as it relies on the fact that the sphere and the torus have nonnegative Euler numbers.

Theorem 6. *Let G be a connected graph which can be embedded on a torus. Then $b(G) \leq \Delta(G) + 3$.*

Proof. Suppose that G is a graph which can be embedded on a torus. By Lemma 3, if G has any vertices of degree 4 or less, we have $\delta(G) \leq 4$, and Theorem 5 holds, so we can assume $\Delta(G) \geq \delta \geq 5$. For the sake of contradiction, assume $b(G) \geq \Delta(G) + 4$. Using the notation of the previous proof, we should have $\sum(v_i + f_i - 1) = |V(G)| + |F(G)| - |E(G)| = 0$, by Lemma 4.

For each i, consider the quantity $v_i + f_i - 1$. If either $d(x)$ or $d(y)$ is equal to 5, by Lemma 3 the other must be equal to $\Delta \geq 5$ and x and y can have no common neighbors, so a_1 and a_2 are both at least 4. This yields $v_i + f_i - 1 < 0$. Suppose one of $d(x)$ and $d(y)$, without loss of generality $d(x)$, is equal to 6. If $d(y) = \Delta - 1 = 5$, then we refer to the previous case. Otherwise, we have $d(y) = \Delta \geq 6$, and at most one of a_1 and a_2 equal to 3, so again we have $v_i + f_i - 1 < 0$. The only remaining case is $d(x), d(y) \geq 7$, but as $a_1, a_2 \geq 3$ in any case we again obtain $v_i + f_i - 1 < 0$. But then summing over all i yields $\sum(v_i + f_i - 1) < 0$, a contradiction to Euler's Formula. □

We can use a similar technique to prove that $b(G) \leq 8$, a result also due to Kang and Yuan [6]. We will employ the following lemma in addition to those above.

Lemma 7 (Hartnell and Rall [5] and Teschner [7]). *If u and v are two vertices of a graph G with $d(u, v) \leq 2$, then*

$$b(G) \leq d(u) + d(v) - 1.$$

Theorem 8 (Kang and Yuan [6]). *If G is a connected planar graph, then $b(G) \leq 8$.*

Proof. Suppose we have $b(G) \geq 9$; we note that by Lemma 3, each edge xy must have $d(x)+d(y) \geq 10$. Using the same notation as before, consider for each edge e_i the quantity $v_i + f_i - 1$. In calculating our f_i, we will disregard any pendant edges in determining the number of sides of a face. For example, we will consider a triangle with a pendant edge in the middle to have three sides, not five. Furthermore, if e_i is an edge with one endpoint of degree 1, we will set $a_1 = a_2 = \infty$ and $f_i = 0$.

Because of the proscription of Lemma 3, the only allowable quadruples $(d(x), d(y), f_1, f_2)$ with $v_i + f_i - 1 > 0$ (up to interchange of x and y and a_1 and a_2) are as follows:

$(1, k, \infty, \infty)$, where $k \geq 9$, and $0 < v_i + f_i - 1 \leq \frac{1}{9}$;

$(2, k, 3, 4)$, where $k \geq 9$, and $0 < v_i + f_i - 1 \leq \frac{7}{36}$;

$(3, k, 3, 3)$, where $k \geq 9$, and $0 < v_i + f_i - 1 \leq \frac{1}{9}$;

$(3, k, 3, 4)$, where $k = 8, 9, 10$, or 11, and $0 \leq v_i + f_i - 1 < \frac{1}{24}$;

$(4, k, 3, 3)$, where $k = 8, 9, 10$, or 11, and $0 \leq v_i + f_i - 1 < \frac{1}{24}$; and

$(5, 7, 3, 3)$, where $v_i + f_i - 1 = \frac{1}{105}$.

We will call such edges **problem edges** and let $P(G)$ be the set of problem edges in G. For each vertex x, we define

$$\alpha(x) \quad = \sum_{\substack{e_i = xy \in P(G) \\ d(x) > d(y)}} (v_i + f_i - 1) \quad + \sum_{e_i = xy \notin P(G)} \frac{1}{2}(v_i + f_i - 1).$$

Now, applying Euler's Formula, we should have

$$\sum_{v \in V(G)} \alpha(v) \quad = \sum_{e_i \in E(G)} (v_i + f_i - 1) = 2.$$

However, we claim that the sum $\alpha(x)$ at each vertex is nonpositive. Clearly, if a vertex x has no problem edges, then $\alpha(x) \leq 0$. Now, when a vertex x has a problem edge assigned to it, we have $d(x) \geq 7$.

If $d(x) = 7$, then each problem edge must be of the form $(5, 7, 3, 3)$, with $v_i + f_i - 1 = \frac{1}{105}$. The endpoints of each problem edge share two neighbors u and v, whose degrees must be at least 7, by Lemma 3. The edges xu and xv have values $v_i + f_i - 1 \leq \frac{-1}{21}$, so each contributes at most $\frac{-1}{42}$ to $\alpha(x)$. Since there is at least one of these edges for each problem edge, we obtain $\alpha(x) < 0$.

If $d(x) \geq 8$, then we have at most one problem edge, since each problem edge in this category has an endpoint of degree at most 4, and having two such vertices at distance two would imply $b(G) \leq 7$, by Lemma 7.

When x has one neighbor y of degree 1, 3, or 4, it must have at least seven neighbors each of degree at least 6, by Lemma 7. Each of the edges between x and a high degree neighbor y satisfies $v_i + f_i - 1 \leq \frac{1}{24}$, since $d(x) = 8$, $d(y) \geq 6$, and $a_1, a_2 \geq 3$. Since none of these are problem edges, they each contribute half their values to $\alpha(x)$. Our problem edge contributes at most $\frac{1}{9}$ to $\alpha(x)$, so we obtain $\alpha(x) \leq \frac{1}{9} - \frac{7}{48} < 0$.

If our problem edge has an endpoint y of degree 2, then $d(x) \geq 9$. Applying Lemma 7, it must have at least eight neighbors each of degree at least 8. Each of the corresponding edges to x contributes at most $\frac{-7}{144}$ to $\alpha(x)$, and xy contributes at most $\frac{7}{36}$. So, $\alpha(x) \leq \frac{7}{36} - \frac{7}{18} < 0$.

□

Here we note that it is sufficient to prove these results for connected graphs, since the bondage number of a disconnected graph is simply the minimum of the bondage numbers of its components.

3. Some planar graphs with bondage number 6

It is not known whether Theorem 8 is tight. In fact, there were previously no known examples of planar graphs with bondage number greater than 5. Here we use the corona graph operation to demonstrate a class of planar graphs with bondage number 6.

In [4], Frucht and Harary define the *corona* of two graphs G_1 and G_2 to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the *i-th* vertex of G_1 is adjacent to every vertex in the *i-th* copy of G_2. In particular, we are concerned with the corona $G = H \circ K_1$, the graph formed by adding a new vertex v_i and the pendant edge $u_i v_i$ for every vertex u_i in H.

Theorem 9. *Let G be a graph of the form $G = H \circ K_1$. Then $b(G) = \delta(H) + 1$.*

Proof. Let $\{u_i\}$ be the vertices of H and $\{v_i\}$ be the corresponding vertices added in the construction of the corona. That is, for each i, the vertex u_i is adjacent to v_i. Then it is clear that $\gamma(G) = |G|/2 = |H|$. In particular, all minimal dominating sets of G are of the following particular form. For each vertex u_i in H, any minimal dominating set of G will contain exactly one of the vertices u_i and v_i.

To increase $\gamma(G)$, we must remove enough edges so that for some i, both u_i and v_i must be in every dominating set on the resulting graph. Clearly, removing any set of edges consisting only of edges in the original graph H will not change $\gamma(G)$, since the pendant edges $u_i v_i$ are the only edges essential for domination, as indicated by the structure of our dominating sets. Thus, we must remove at least one edge $u_i v_i$. Doing so requires us to include v_i in any dominating set on the resulting graph. However, u_i still has $d_G(u_i) - 1 = d_H(u_i)$ neighbors, so we must remove that many edges to force u_i to be in every dominating set. Applying this technique at a vertex of minimal degree in H yields $b(G) = \delta(H) + 1$, as desired. □

Corollary 10. *There exist planar graphs with bondage number 6.*

We note that there exist planar graphs H with $\delta(H) = 5$. Taking the corona $G = H \circ K_1$ gives a planar graph with $b(G) = 6$. One such example is the corona of K_1 and the graph of the icosahedron.

4. A bound on the bondage number of directed graphs

The notion of bondage is equally apt in the case of directed graphs, although to date no research has been done on this concept. We will use the following notation in dealing with directed graphs.

If G is a directed graph, we denote the in-degree of a vertex u by $\overleftarrow{d}(u)$ and its out-degree by $\overrightarrow{d}(u)$. The maximum in-degree (respectively out-degree) of any vertex in G is denoted by $\overleftarrow{\Delta}(G)$ (respectively $\overrightarrow{\Delta}(G)$); the minimum in-degree (respectively minimum out-degree) is denoted by $\overleftarrow{\delta}(G)$ (respectively $\overrightarrow{\delta}(G)$). The directed distance between u and v is denoted by $\overrightarrow{d}(u, v)$. We define $\overleftarrow{N}(v)$ (respectively $\overrightarrow{N}(v)$) to be the set of all vertices u for which there exists an edge $\overrightarrow{uv}$ (respectively $\overrightarrow{vu}$). We define the neighborhood of v to be $N(v) = \overleftarrow{N}(v) \bigcup \overrightarrow{N}(v)$. For a set S of vertices, we set $\overleftarrow{N}(S) = \cup_{v \in S} \overleftarrow{N}(v)$, and similarly for $\overrightarrow{N}(S)$ and $N(S)$. Given an undirected graph G, we define the corresponding directed graph (which we will also call G) by replacing each undirected edge with a pair of directed edges.

For a directed graph G, we say a set D is a *dominating set* if $V(G) - D \subset \overrightarrow{N}(D)$. Then, just as for undirected graphs, $\gamma(G)$ is the minimum size of a dominating set, and $b(G)$ is the smallest size of a set of edges E such that $\gamma(G - E) > \gamma(G)$.

Lemma 3 allows us to calculate an upper bound on bondage number, which is a global property based on local properties of the graph. It has the following natural extension to directed graphs, with a proof essentially identical to that of Hartnell and Rall [5].

Lemma 11. *If G is a directed graph, then for every pair of vertices (u, v) in G for which there exists an edge from u to v,*

$$b(G) \leq d(v) + \overleftarrow{d}(u) - |\overleftarrow{N}(u) \cap \overleftarrow{N}(v)|.$$

Proof. Consider the set T consisting of all edges incident to v and all edges terminating at u. From this set, remove those edges $\overrightarrow{wu}$ for which the edge $\overrightarrow{wv}$ also occurs in G, and let S be the resulting set. By construction, $|S| = d(v) + \overleftarrow{d}(u) - |\overleftarrow{N}(u) \cap \overleftarrow{N}(v)|$. To prove the lemma, we need only to show that $\gamma(G - S) > \gamma(G)$, or, equivalently, that no minimal dominating set for G can also dominate $G - S$. Suppose that D were such a set. Then as v is isolated in $G - S$, the set D must contain v. Now, either $u \in D$ or $w \in D$ with $\overrightarrow{wu}$ existing in $G - S$; by choice of S, either of these conditions implies $v \in \overrightarrow{N}(D - \{v\})$ in G. As v is an isolated vertex in $G - S$, we have $V(G) - \{v\} \subset (D - \{v\}) \cup \overrightarrow{N}(D - \{v\})$ in $G - S$ and therefore G. But this means that $D - \{v\}$ is a dominating set for G, contradicting the assumption that D is a minimal dominating set for G. □

Lemma 11 then yields the following immediate corollary.

Corollary 12. *If G is a directed graph, $b(G) \leq \overleftarrow{\delta}(G) + \Delta(G)$.*

As we have $\overleftarrow{\delta}(G) \leq \frac{1}{2}\Delta(G)$, this in turn yields the desired result, proving Conjecture 2 in the case of directed graphs.

Corollary 13. *If G is a directed graph, $b(G) \leq \frac{3}{2}\Delta(G)$.*

Unlike the case of undirected graphs, however, it is no longer clear whether or not the bound in Corollary 13 is sharp. Consider the aforementioned family of graphs $G_n = K_n \times K_n$, which establishes the sharpness of Conjecture 2 in the case of undirected graphs. If we take the corresponding directed graph, we have $\Delta(G_n) = 4(n - 1)$; the domination number of G_n is still n. However, label the vertices of K_n with the set $\{0, 1, \ldots, n-1\}$, and consider the set S consisting of all edges terminating at $(0, 0)$, all edges from $(0, 0)$ to $(0, j)$, all edges from $(1, j)$ to $(1, 0)$, and the single edge from $(0, 0)$ to $(1, 0)$. Clearly, any dominating set D for $G_n - S$ must contain $(0, 0)$ as well as some other $(i, 0)$. Furthermore, $(G_n - S) - \{(0, 0), (i, 0)\} - \overrightarrow{N}(0, 0) - \overrightarrow{N}(i, 0)$ is isomorphic to G_{n-1} and thus has dominating number $n-1$, and it is easy to see that we cannot find a set of fewer than $n-1$ elements which dominates this set in the larger graph $G_n - S$. Therefore, D must contain at least $n + 1$ elements, so $\gamma(G_n - S) > \gamma(G_n)$. By definition, we then have $b(G_n) \leq |S| = 4(n - 1) + 1$. In particular, this family of graphs does not prove that Corollary 13 is sharp.

Indeed, the following conjecture, discredited in the case of undirected graphs, is resurrected here in the case of directed graphs.

Conjecture 14. *If G is a directed graph, then $b(G) \leq \Delta(G) + 1$.*

This bound, if true, is shown to be sharp by the same class of examples as in the undirected case. Specifically, we have the following result.

Theorem 15. *Let K_n be the directed complete graph on n vertices, and let $G_n = K_n \times K_n$. Then $b(G_n) = 4(n - 1) + 1 = \Delta(G_n) + 1$.*

Proof. We have already shown that $b(G_n) \leq 4(n-1)+1$. To show the reverse implication it suffices to show that for every set $S \subset E(G_n)$ with $|S| = 4(n-1)$, there exists some dominating set of $G_n - S$ with size n. We call the set $\{i,k\}$ for constant i and variable k the $\mathbf{i}^{th}$ **row** of G_n, and $\{k,j\}$ for constant j and variable k the $\mathbf{j}^{th}$ **column** of G_n. We divide the edge set of G_n into **column edges**, which connect two elements of the same column, and **row edges**, which connect two elements of the same row. For a given set S, we say that a vertex v **dominates** its row (respectively, column) if none of the row edges (respectively, column edges) emanating from v are in S. We call a row (respectively, column) a **problem row** (respectively, **problem column**) if it contains no vertex dominating it. Note that each vertex in such a row or column must have at least one row or column edge emanating from it in S, and so each problem row or column must have n corresponding row or column edges in S.

Fix S with $|S| = 4(n-1)$. Without loss of generality, assume S contains at most $2(n-1)$ column edges. If each column has a dominating vertex, we can take the union of these vertices to form a dominating set for $G_n - S$ of size n. Otherwise, S has exactly one problem column; assume without loss of generality that this is the 0^{th} column.

Now, there exists some $(k,0)$ with only one column edge emanating from it in S, as $|S| < 2n$. Without loss of generality let this edge go from $(1,0)$ to $(0,0)$. If there exists some j for which $(0,j)$ both has an edge to $(0,0)$ and dominates its column, we can pick the set $\{(1,0),(0,j),v_t)\}$ where v_t dominates column t, $t \neq 0, j$; this set will then dominate $G_n - S$ and be of size n. Therefore, we can assume that for every j, either the edge from $(0,j)$ to $(0,0)$ or some column edge emanating from $(0,j)$ is in S. This specifies an additional $n-1$ edges of S. We now consider the column edges emanating from $(0,0)$.

Claim 16. *Suppose that there are m column edges in S emanating from $(0,0)$; let $\{k_i\}$ be the set of rows containing a terminal vertex of one of these edges. Then one of the following two statements must hold:*

(i) We can find $n-m$ edges of S and a row $i \neq 0$ such that each edge is either a column edge or lies in row i and terminates at $(i,0)$, or

(ii) We can find distinct $j_i \neq 0$ such that (k_i, j_i) both has an edge to $(k_i, 0)$ and dominates its column in $G_n - S$.

Proof. The proof is by induction. If $m = 1$, then unless (ii) is true, for each j we have emanating from (k_1, j) either an edge to $(k_1, 0)$ or a column edge in S; since this must be the case for $1 \leq j \leq n-1$, we obtain $n-1$ edges all either column edges or contained in row k_1.

If $m > 1$, then we apply the claim to the first $m-1$ of these edges. If (i) is true, we can find $n-m+1$ edges of S satisfying the constraint in question, so we can certainly find $n-m$ edges satisfying it. If (ii) is true, consider all $j \neq 0, j_1, \ldots, j_{m-1}$. If any j dominates its column and has an edge to $(k_m, 0)$, we can set $j_m = j$ and satisfy condition (ii). If this is not the case, then for any j either the edge from (k_m, j) to $(k_m, 0)$ lies in S or some column edge containing (k_m, j) lies in S, and we can take the subset of S consisting of all such edges. This set has cardinality $n-m$ and contains only column edges or edges in row k_m terminating at $(k_m, 0)$, hence satisfies condition (i) as desired. □

If (ii) holds, then we can take our dominating set to be $\{(0,0),(k_i,j_i),v_t\}$ where i ranges from 1 to m and the v_t dominate column t, $t \neq 0, j_i$.

On the other hand, (i) implies the existence of $n-m$ edges which are either column edges not in column 0 or contained in a given row i. Furthermore, we know now that S contains at least $n+m-1$ edges in column 0, for the original count of n included only one edge emanating from each vertex and we now know that m edges emanate from $(0,0)$. This brings our total number of edges shown to be in S up to $3n-2$. Note that these edges are all either in column 0 or have initial vertex in row 0 or row i and not in column 0.

Now, let l be the total number of row edges in S emanating from $(0,0)$ and $(i,0)$. The proof of the following claim is identical to that of Claim 16, and will be omitted.

Claim 17. *Suppose there are l row edges in S, each emanating from $(0,0)$ or $(i,0)$; let $\{k_t\}$ be the set of all columns containing a terminal vertex of one of these edges. Then one of the following two statements holds:*

(i) We can find a set of $n-l-1$ edges in S not previously enumerated, or

(ii) We can find distinct $j_t \neq 0, i$ such that (j_t, k_t) has edges to $(0, k_t)$ and (i, k_t) and dominates its row in $G-S$.

However, (i) is impossible, as adding these edges and the l row edges in S to the previously enumerated edges of S yields $|S| \geq 4n-3$, a contradiction. Furthermore, if (ii) is true, the set $\{(0,0),(i,0),(j_t,k_t),v_s\}$ dominates S, where v_s dominates row s, $s \neq 0, i, j_t$. Note that such v_s must exist, as for the specified values of s, the previously enumerated $3n-2$ edges of S contain no edges of row s, so that only $n-2$ edges of row s can occur in S and row s cannot be a problem row. □

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