A Study of Model-Order Reduction Techniques for Verification

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Abstract. As formal verification techniques for cyber-physical systems encounter large plant models, techniques for simplifying these models into smaller approximate models are gaining increasing popularity. Model-order reduction techniques take large ordinary differential equation models and simplify them to yield models that are potentially much smaller in size. These approaches typically discover a suitable projection of the state space into a smaller subspace, such that by projecting the dynamics in this subspace, an accurate approximation can be obtained for a given initial set and time horizon of interest. In this paper, we present a study of model-order reduction techniques for verification with non-rigorous error bounds. We design experiments based on the proper orthogonal decomposition technique for finding reduced order models. We find that reduced order models are particularly effective and precise whenever a suitable reduced order model can be found in the first place. We attempt to characterize these models and provide future directions for reduced order modeling.

1 Introduction

In this paper, we study the potential application of model order reduction to reachability analysis of nonlinear systems. Even though significant progress has been made for linear systems [3,10], the number of state variables remains a significant limitations for applications to nonlinear systems. In this context, model order reduction techniques that approximate a large system by a reduced order system over a smaller state space have been promising [12,22]. Studies on using model order reduction of the verification of linear systems have been carried out recently [22]. Here, we report on a systematic study of model order reduction for nonlinear systems.

Our study implements a popular model order reduction approach called *proper orthogonal decomposition* (POD), that reduces a large model using a singular value decomposition (SVD) of a snapshot matrix constructed from states obtained through numerical simulations [5,18]. The SVD yields a projection from the original state space to a reduced state space over which we define reduced order ODE model. This model is often promised to be much smaller (eg. 3 state variables) when the original model is much larger (eg. 500 state variables). Hence, we study the applicability of model-order reduction techniques for reachability

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computation. More precisely, we try to reduce the size of a large-scale system and then prove its safety by reachability computation, that is to prove that the overapproximation set has no intersection with the unsafe set. However, doing so forces us to work with a lower dimensional approximation, whose behaviors are empirically seen to be *close to* the original system. Therefore, the approach loses formal guarantees wrt the original model. Nevertheless, these approximations can be justified provided the error can be systematically estimated and it is understood that the original model itself is an approximation of a more complex "physical reality".

We perform an experimental evaluation using the Flow* tool to compute reachable sets over a finite time horizon using the original model (wherever possible) and the reduced order models. We compare the flowpipes so obtained against safety properties that are checked on both models. Next, we examine the effect of varying initial set sizes and time horizons for both models. We summarize the major findings of our experimental evaluation as follows:

1. Model order reduction approaches are not universally applicable. In fact, whether or not they will yield a significant reduction in the size of the model depends on how the singular values obtained during POD decay. On some systems, a rapid exponential decay yields significant reductions. However, such decays are not seen in other systems where model order reduction simply fails to yield a significant reduction or the reduced model is no easier to verify than the original.
2. Whenever a significant reduction is possible, we observe that using careful choice of the reduced model size, we obtain promising small models that are also very close to the original system. This shows that model order reduction can be considered a good approach to perform reachability analysis of large models that are otherwise out of the reach of existing approaches.
3. The performance varies, depending on the sizes of the initial sets and time horizon of interest, and also depending on the sensitivity of the original model to these parameters.

1.1 Related Work

Model order reduction (MOR) obtains a lower-dimensional approximation of a given system such that the trajectories of the approximate system are as close as possible to the original system over a set of initial conditions and time horizon. As a result, MOR is potentially a useful approach to solve control synthesis and verification problems [11,16]. A variety of approaches for MOR for linear systems exist, such as balanced truncation (BT) [16] and Krylov Methods (KMs) [20]. For nonlinear systems, the most common MOR approaches include Proper Orthogonal Decomposition (POD) and Trajectory Piecewise Linearization (TPWL). POD method calculates the singular value decomposition of *snapshot matrix* generated from trajectories sampled at certain time instants [18]. On the other hand, TPWL transforms the original nonlinear system into a combination of a set of reduced-order linear models by a process of hybridization around specific

trajectory states [19]. The POD approach is used as a basis for this work, and will be described further in Sect. 3.

MOR has been explored for solving linear system verification problems [12, 13,22]. Han and Krogh developed a reachability approximation procedure with the balanced truncation reduction method, and they analyzed the error bound derived from reducing order for verification of high-dimensional linear systems [12]. They noticed that large error bounds can be avoided for initial conditions that are close to the equilibrium points of these systems. Han and Krogh extend this subsequently to consider Krylov Methods for analyzing reachability problems of large-scale affine systems [13]. Recently Tran et al. extend this work by comparing several methods for estimating error bounds in several high-dimensional linear system benchmarks [22].

However, for nonlinear systems, it is hard to estimate a conservative error bound for the reduced order model analytically. This forces us to rely in this paper on statistical techniques that do not guarantee an error bound. These techniques are justified if we can empirically show that the MOR approach approximates the model to a reasonable degree of accuracy. Since the model is often itself an approximation of the underlying physical reality, an accurate low dimensional approximation is justified. To this end, we also check if the reduced order model can preserve properties when compared to the original model.

2 Preliminaries

In the paper, we use $\mathbb{R}$ to denote the set of real numbers. Given a set of ordered variables $x_1, \ldots, x_n$, we collectively denote them by a column vector $\boldsymbol{x}$. For a column vector $\boldsymbol{x}$, its ith component is denoted by x_i. Given a variable x, we use $\dot{x}$ to denote its time derivative dx/dt.

A (nonlinear) system is defined by an ODE $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, wherein $\boldsymbol{x}$ represents the n state variables. The function f is assumed to be at least *locally Lipschitz continuous*, and therefore there is a unique solution in an open neighborhood of a given initial state $\boldsymbol{x}(0) \in \mathbb{R}^n$ for the ODE. A *state* of the system at time t, is denoted by the vector $\boldsymbol{x}(t)$.

Since most nonlinear ODEs do not have solutions expressed in a closed form, we focus instead on computing approximations of their solutions. Given an ODE $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ along with an initial condition $\boldsymbol{x}(0) \in X_0$. The exact solution is denoted by $\boldsymbol{x}(t) = \varphi_f(\boldsymbol{x}_0, t)$. This can either be numerically approximated using numerical integration approaches, or an overapproximate reachable set can be computed using numerous tools, such as SpaceEx for linear ODEs [10] and Flow* for nonlinear ODEs [7].

Flowpipe construction techniques compute reachable set overapproximation segments over a bounded time horizon. That is, each segment contains the exact reachable states over a small time interval which is also called a time step (see [1,2,4,6,9,17]). These methods are usually based on a flowpipe construction framework, that is, a reachable set segment (flowpipe) is computed based on the previous one.

3 Proper Orthogonal Decomposition

In this section, we describe the process of computing a reduced order model. We mainly focus on the method called *Proper Orthogonal Decomposition (POD)* [5, 18,21], which projects a large-scale system to a much smaller one using a linear mapping. The linear projection is computed using singular value decomposition (SVD). The approach is also called *Principal Component Analysis (PCA)* in other areas [15].

Let $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ be an autonomous nonlinear system. Furthermore, let X_0 be an initial region of interest and $t \in [0, t_f]$ be a fixed time horizon over which the system is to be approximated. The POD approach computes a matrix $n \times k$ projection matrix U_r where $k \ll n$, such that the new state variables are given by $\boldsymbol{z} := U_r^T \boldsymbol{x}$. Further, we require the columns of U_r to be orthonormal: $U_r^T U_r = I$. Although U_r is not invertible, we write $\tilde{\boldsymbol{x}} := U_r \boldsymbol{z}$ as the "representative" state for a given reduced state $\boldsymbol{z}$. Therefore, we translate the dynamics as

$$\dot{\boldsymbol{z}} = U_r^T \dot{\boldsymbol{x}} = U_r^T f(\boldsymbol{x}) \simeq U_r^T f(U_r \boldsymbol{z}) \,.$$

The matrix U_r is discovered as follows:

1. Randomly select an initial value $\boldsymbol{x}_0 \in X_0$ and numerically simulate the time trajectory to obtain a *snapshot matrix* X using a step size $\delta > 0$:

   $$X : [\boldsymbol{x}(0), \boldsymbol{x}(\delta), \ldots, \boldsymbol{x}(N\delta)] \text{ wherein } N\delta = t_f$$

   The columns of the snapshot matrix are the states of the system at times $0, \delta, \ldots, N\delta$ for a chosen time step δ. X is thus a matrix with n rows and N columns.
2. Use singular value decomposition (SVD) to factor X as

   $$X = UDV^T, \text{ wherein } U \in \mathbb{R}^{n \times n},\ D \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{N \times n} \,.$$

   SVD guarantees that $UU^T = I$ and $V^T V = I$. Furthermore, D is a diagonal matrix of the singular values of X. Let the singular values be $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.
3. Choose a number k and the corresponding k largest singular values $\sigma_1 \geq \cdots \geq \sigma_k$. The choice of this k dictates the approximation error, and is discussed subsequently.
4. The projection matrix U_r is given by the k columns of U corresponding to the chosen singular values.

Lemma 1. *U_r is the $n \times k$ matrix that is the minimizer of the error function:*

$$\min_{P \in \mathbb{R}^{n \times k}} \|X - PP^T X\|$$

wherein $\|\cdot\|$ denotes the Frobenius norm. Furthermore, choosing $P = U_r$ provides the optimal value $\sum_{i=k+1}^{n} \sigma_i^2$ equal to the sum of the squares of the truncated singular values.

One of the main assumptions that govern the applicability of this approach is that the singular values of the snapshot matrix "decay" rapidly so that

$$\sigma_{i+1}, \ldots, \sigma_n \simeq 0.$$

As a result retaining $k \ll n$ singular values is sufficient for approximating the system to a high degree, and with very low error. Note however, that if the singular values do not decay in this manner, the approach can fail to yield a sufficiently precise and small reduced order model.

3.1 An Illustrative Example

As a simple example, consider the following 3 state nonlinear system modeling three coupled oscillators. The dynamics are given as

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{x} - \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix} + Bu, \text{ and } x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

where $B = [1, 0, 0]^T$, and input u is set to the constant 1. We simulate the system within time horizon $[0, 2]$ to construct the snapshot matrix X, using step size $\delta = 0.1$. By solving the SVD for X, we obtain three left singular vectors $U_1 = [-0.9253 \quad -0.3606 \quad -0.1171]^T$, $U_2 = [0.3706 \quad -0.7948 \quad -0.4805]^T$, and $U_3 = [0.0802 \quad -0.4881 \quad 0.8691]^T$, with the corresponding singular values $\sigma = [5.9670 \; 0.5495 \; 0.0508]$. Thus, U_1 dominates the system's behavior, which allows to project the system into a 1-dimension subspace. With the projection $U_r^T = U_1^T$, the reduced order ODE equation becomes

$$\dot{z} = U_r^T f(U_r z) + U_r^T B u \tag{2}$$

$$= 0.8408z^2 - 1.2482z - 0.9253u, \tag{3}$$

$$\text{with } z(0) = U_r^T \boldsymbol{x}(0) \tag{4}$$

After simulating the trajectories of the reduced model, the approximated trajectory can be obtained by projecting back to the full-order space with the relation $\boldsymbol{x}_r = U_r z$. Figure 1, shows the simulations for the original model vs the reduced order model for the state variable $x_1(t)$.

However, the situation considered for verification is complicated by the choice of a random $\boldsymbol{x}_0 \in X_0$. Ideally, by obtaining a good approximation for a single state $\boldsymbol{x}_0$, we expect that the reduced order model continues to serve as a good approximation for states that are "near" $\boldsymbol{x}_0$. Nevertheless, there are no guarantees that this is the case in practice. Therefore, we now focus on the choice of the cutoff k to guarantee an empirical error estimate over X_0 and $[0, t_f]$.

3.2 Error Estimation and Control

As mentioned earlier, the choice of k dictates the size vs accuracy tradeoff for the reduced order model. Typically, one can select the value k such that $\frac{\sum_{i=k+1}^{n} \sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2} < \kappa$

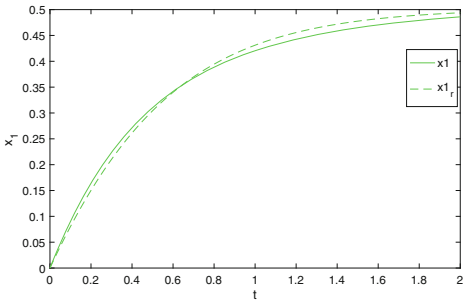


Fig. 1. Comparing the time evolution of x_1 of the original system against the approximation x_{1r} obtained from the reduced order model.

for a user-specified tolerance κ (often set as 0.01 [18]). However, the proper tolerance κ varies with different systems and furthermore, since the snapshot is constructed for one initial state $\boldsymbol{x}_0$, the singular values σ_i do not necessarily predict the behavior for other initial states. Therefore, we measure a simulation error metric by sampling a fixed number K of states from the initial region X_0 and comparing the reduced order model with the original model:

$$e_s = \max \left\{ \frac{\|\boldsymbol{x}_s - U_r U_r^T \boldsymbol{x}_s\|}{\max\{1, \|\boldsymbol{x}_s\|\}} \;\middle|\; \boldsymbol{x}_s \text{ is a simulation sample} \right\}$$

wherein a simulation sample means a state in a (discrete) simulation trace.

Then, we choose a value of k to satisfy that the maximum error of the sampling states is smaller than simulation tolerance 0.05 in our study. If no such value can be obtained, the process of simulation is said to fail.

In order to take the size of an initial set into account in MOR, we consider to bloat the unsafe set in a reduced state space by adding a statistical error bound which is described in [14]. Although the error bound is not rigorous, it still reflects the quality of reduction.

4 Experiments

We implemented the ideas presented so far to compute a reduced order model for a given system of ODEs. In our implementation, the time horizon $[0, t_f]$, range of initial values for each variable $[\boldsymbol{x}_{0,min}, \boldsymbol{x}_{0,max}]$, the number of random initial simulation points l, and time step δ used in snapshot matrix are given as parameters.

Using the POD algorithm introduced previously, we have a k-order reduced model based on a projector U_r, which has a simulation error below a specified tolerance ($\delta_s = 0.05$). Next, the initial condition Z_0 for the reduced subsystem is obtained by computing $U_r^t X_0$ and then overapproximating it using a box: $[\boldsymbol{z}_{0min}, \boldsymbol{z}_{0max}]$. Furthermore, safety properties are bloated using the error bound $|\delta_s|$, the transformed unsafe region is also over-approximated. We use the tool

Flow* to construct the flowpipe for the reduced order model and compare this with the flowpipe for the original model whenever we are able to compute these.

In this section, we investigate the reachability of unsafe specification in several nonlinear systems with the verification technique introduced early, and further their computation will be evaluated. We first describe the benchmarks used wherein the method was successful, the reduced models for these benchmarks and flowpipes computed using the Flow* tool.

4.1 Description of Benchmarks and Results

We first present a description of some of the benchmarks and the results obtained for each benchmark.

Analog Circuits. The first circuit example is a nonlinear transmission line circuit model considered in Ref. [8], and the ODE is written as $\frac{dx}{dt} = f(x) + Bu$, wherein

$$f(\boldsymbol{x}) = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 - e^{0.1x_1} - e^{0.1(x_1 - x_2)} \\ e^{0.1(x_1 - x_2)} - e^{0.1(x_2 - x_3)} \\ \vdots \\ e^{0.1(x_{n-2} - x_{n-1})} - e^{0.1(x_{n-1} - x_n)} \\ e^{0.1(x_{n-1} - x_n)} - 1 \end{bmatrix}, \quad (5)$$

$B = [1, 0, \cdots, 0]^T$, and input $u = 0.5$ is current source entering node. The state vector $x = [x_1, x_2, \cdots, x_n]^T$ models the voltages at n points in the transmission line. For simplicity, we set all the resistors and capacitors as unit resistance and capacitance. Here, we only consider the output variable as x_1, the voltage at the first node.

Here, we consider $n = 100$, and POD reduced method can greatly decrease its dimension to $k = 2$. The initial region is

$$X_0 : \ \{x \,|\, x_i \in [0, 0.0025] \text{ for } i = 1, \ldots, 20 \text{ and } x_i = 0 \text{ for } i = 21, \ldots, 100\}$$

and the time horizon $t_f = 3$. The simulations of the original and reduced order systems are shown in Fig. 2 in solid and the dotted lines, respectively. The simulations are shown both in the original state space $\boldsymbol{x}$ and the reduced order state space $\boldsymbol{z}$ obtained by projection.

Furthermore, we compute the flowpipe segments using the Flow* tool, and the result is shown in Fig. 3. Figure 3(a) represents the flowpipe computed for the reduced order variable z_1 which contain the simulation trajectory in Fig. 2(b), and in Fig. 3(b) we further compare it with the flowpipe of the original system, also computed using Flow*. Note that in order to project the flowpipe from a low-dimensional subspace to a high-dimensional space, we choose roughly 10^4 samples of $\boldsymbol{z}$ from the flowpipe and apply the matrix U_r to project them back so that the high dimensional flowpipe can be visualized.

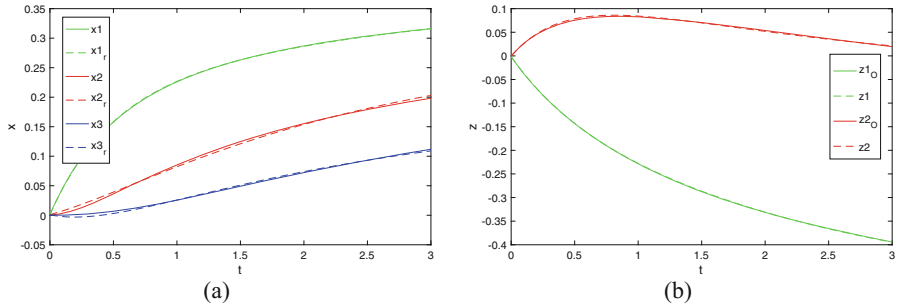


Fig. 2. A nonlinear transmission line circuit model: (a) Simulation trajectories of the original system and the reduced system in full-dimensional space. (b) Simulation trajectories of the original system and the reduced system in reduced-dimensional subspace

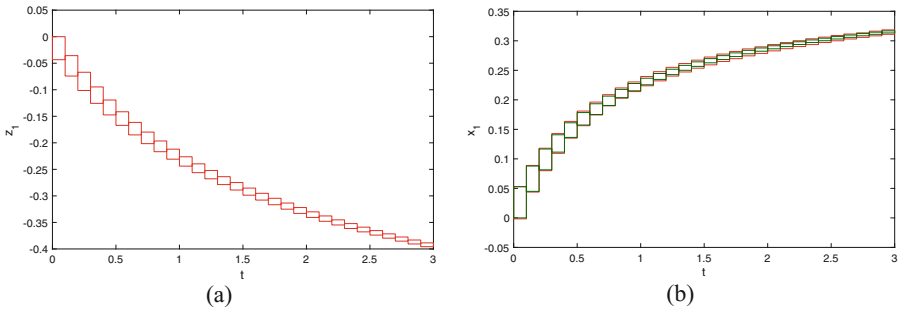


Fig. 3. Flowpipes for the nonlinear transmission line circuit model: (a) reduced model in the reduced state space, (b) comparison between the reduced model and the original model in the original state space.

The original set of unsafe states are defined as $X_f(S_n) = \{\boldsymbol{x} \,|\, x_1 > 0.35\}$. This is relaxed to the region $X_f(S_n) = \{\boldsymbol{x} \,|\, x_1 > 0.29\}$, taking into account a statistical error bound $|\delta_s| = 0.06$. It is further transformed to the property

$$X_f(S_k) = \{\boldsymbol{z} \,|\, U_r(1,1)z_1 + U_r(1,2)z_2 > 0.29\}$$

over the reduced subspace. From Table 1, this safety property is not satisfied in the reduced model, whereas it is satisfied in the original model.

Next, we modify the transmission line circuit model with a quadratic nonlinearity by adding a nonlinear resistor to the ground at each node, and thus the RHS of the ODE now becomes [19]:

$$f(\boldsymbol{x}) = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \boldsymbol{x} - \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{n-1}^2 \\ x_n^2 \end{bmatrix}, \tag{6}$$

where $B = [1, 0, \cdots, 0]^T$, input $u = 1$, $n = 100$, the initial region

$$X_0 = \{\boldsymbol{x} \mid x_i \in [0, 0.0025] \text{ for } i = 1, \ldots, 20 \text{ and } x_i = 0 \text{ for } i = 21, \ldots, 100\}$$

and time horizon $t_f = 3$. In this example, POD method can reduce the dimension to $k = 2$. Figure 2 shows the simulation trajectories, and Fig. 5 shows the flowpipes (Fig. 4).

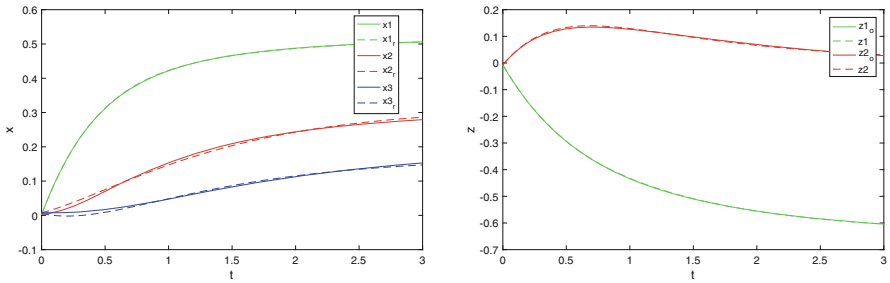


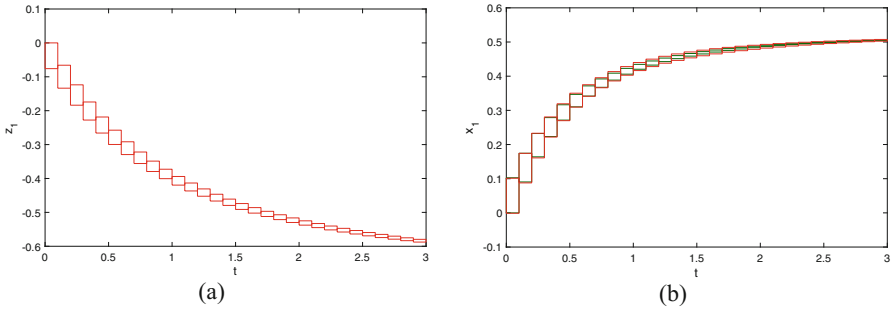
Fig. 4. A nonlinear transmission line circuit model with quadratic nonlinearity: (a) Simulation trajectories of the original system and the reduced system in full-dimensional space. (b) Simulation trajectories of the original system and the reduced system in reduced-dimensional subspace.

4.2 Fluid Dynamics

In this benchmark, we explore a fluid dynamics model for shock movement in a fluid obtained by discretizing a 1-dimensional Burgers PDE model:

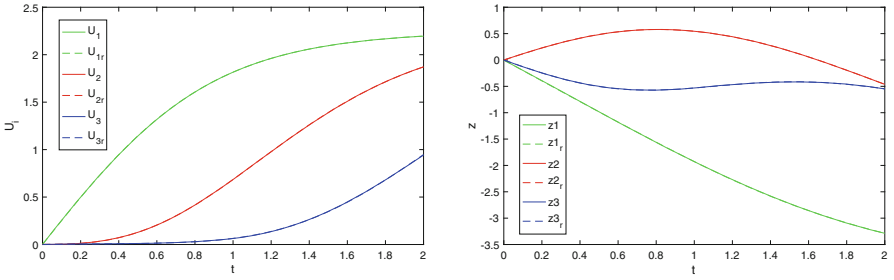
$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial F(U(x,t))}{\partial x} = G(x), \quad \text{with } U(x,0) = 1,\ U(0,t) = u(t), \tag{7}$$

We assume that $x \in [0, l]$, where l is the length of the modeled region, U is a conserved quantity (eg. density, heat), $F(U) = 0.5U^2$, $G(x) = 0.02\exp(0.02x)$, and $u(t)$ is the incoming flow. After discretizing the partial differential equation with $\bar{U} = [U_1, U_2, ..., U_n]^T$, where U_i approximates U at point $x_i = i\Delta x$ ($\Delta x = \frac{l}{n}$, where n is the number of grid points), we have an n-dimensional ODE equation shown below [19]:



(a) (b)

Fig. 5. Flowpipes for the quadratic nonlinearity model: (a) reduced model in the reduced state space, (b) comparison between the reduced model and the original model in the original state space.



(a) (b)

Fig. 6. Fluid dynamic model: (a) Simulation trajectories of the original system and the reduced system in full-dimensional space. (b) Simulation trajectories of the original system and the reduced system in reduced-dimensional subspace.

$$\frac{d\bar{U}}{dt} = f(\bar{U}) + Bu, \tag{8}$$

$$f(\bar{U}) = \frac{1}{\triangle x}\begin{bmatrix} -0.5U_1^2 & & & \\ & 0.5(U_1^2 - U_2^2) & & \\ & & \ddots & \\ & & & 0.5(U_{n-1}^2 - U_n^2) \end{bmatrix} + \begin{bmatrix} e^{0.02x_1} \\ e^{0.02x_2} \\ \vdots \\ e^{0.02x_n} \end{bmatrix} \tag{9}$$

where $B = [1/(2\triangle x)0 \cdots 0]$, and $u = 5$ in our study. Also, we consider the initial region

$$\bar{U}_0 = \{\bar{U} \,|\, U_i \in [0, 0.0025] \text{ for } i = 1, \ldots, 20 \text{ and } U_i = 0 \text{ for } i = 21, \ldots, 100\}$$

and $t_f = 2$. In this case, the dimension is reduced considerably from $n = 100$ to $k = 3$. Simulating the trajectories, there is no noticable difference between that from full-order model and that from reduced-order model, which is shown in Fig. 6. As we expected, Fig. 7 compares the flowpipes for the reduced model contain that of the original model.

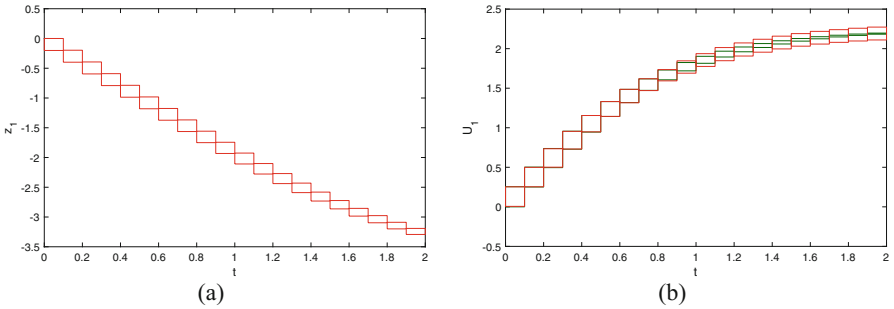


Fig. 7. Flowpipes for the fluid dynamic model: (a) reduced model in the reduced state space, (b) comparison between the reduced model and the original model in the original state space.

Table 1. Checking properties for the original model S_n vs. the reduced order model S_k: for each benchmark, the initial condition region and time region are described in the text.

Benchmark	$X_f(S_n)$	Safe	$X_f(S_k)$	Safe
Analog circuits.1($n = 100$)	$x_1 \geq 0.35$	Yes	$U_r(1,1) * z_1 + U_r(1,2) * z_2 \geq 0.29$	No
	$x_2 \geq 0.25$	Yes	$U_r(2,1) * z_1 + U_r(2,2) * z_2 \geq 0.19$	No
	$x_3 \geq 0.20$	Yes	$U_r(3,1) * z_1 + U_r(3,2) * z_2 \geq 0.14$	Yes
Analog circuits.2($n = 100$)	$x_1 \geq 0.6$	Yes	$U_r(1,1) * z_1 + U_r(1,2) * z_2 \geq 0.54$	Yes
	$x_2 \geq 0.4$	Yes	$U_r(2,1) * z_1 + U_r(2,2) * z_2 \geq 0.34$	Yes
	$x_3 \geq 0.2$	Yes	$U_r(3,1) * z_1 + U_r(3,2) * z_2 \geq 0.14$	No
Fluid dynamics($n = 100$)	$x_1 \geq 2.5$	Yes	$U_r(1,1) * z_1 + U_r(1,2) * z_2 + U_r(1,3) * z_3 \geq 2.426$	Yes
	$x_2 \geq 2$	Yes	$U_r(2,1) * z_1 + U_r(2,2) * z_2 + U_r(2,3) * z_3 \geq 1.926$	Yes
	$x_3 \geq 1.5$	Yes	$U_r(3,1) * z_1 + U_r(3,2) * z_2 + U_r(3,3) * z_3 \geq 1.426$	Yes

4.3 Evaluation

We now discuss the results over the benchmarks in detail. Table 2 presents the details of the benchmarks, the sizes of the reduced order model, the running time for flowpipe computation in the original vs reduced models, and the error bounds for each of the benchmarks used in our example. As expected, we have nonlinear systems as large as 500 variables over which we obtain very small reduced order models that are handled by Flow* whereas the original system cannot be within the given time out for one hour. However, for the P53 model, we observe that Flow* is able to compute the flowpipe for the original model but times out on the reduced order model. There are many reasons: for one the number of state variables is just one indicator for the difficulty of a flowpipe

construction problems. Factors that depend on the behavior of the model's time trajectories also play a critical role.

Figure 8 plots the singular values for the fluid dynamic and the P53 models. In one case, we note the rapid exponential decay in the singular values allowing us to achieve a large reduction, whereas in the other we see that the singular values remain relatively large in magnitude.

Table 2. The computation time and the statistical error bound corresponding to different benchmarks: n is the dimension of a original system, S_n is the reachable set for the original system, $T(S_n)$ is the time in seconds for flowpipe over the original system, k is the dimension of a reduced-order abstraction, S_k is the reachable set for the reduced system, $T(S_k)$ is time taken by flowpipe computation on the reduced model, $|\delta_s|$ is the total statistical error bound, OOT represents "out of time" with the time limit = 1 h.

Benchmark	$T(S_n)$	k	$\|\delta_s\|$	$T(S_k)$
Analog circuits.1($n = 100$)	59	2	0.06	1.5
Analog circuits.1($n = 500$)	OOT	2	0.13	1.8
Analog circuits.2($n = 100$)	498	2	0.06	0.5
Analog circuits.2($n = 500$)	OOT	2	0.13	0.5
Fluid dynamics($n = 100$)	205	3	0.074	1.4
Fluid dynamics($n = 500$)	OOT	3	0.23	2
Cellular P53 regulation($n = 6$)	110	4	51	OOT

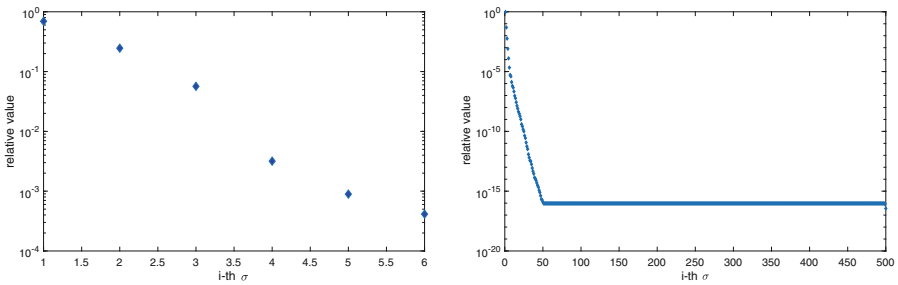


Fig. 8. The singular values obtained for (a) Cellular P53 regulation benchmark. (b) Fluid dynamics benchmark.

Next, we vary the sizes of the initial region for each of the benchmark while continuing to look for a reduced model that conforms to a fixed error bound. Table 3 shows the results. Here, we note that as the initial region is made larger, the time taken to estimate the error bound increases, the size of the model increases and so does the computation time for the flowpipes. In some cases, the model order reduction is no longer successful in obtaining a flowpipe within

Table 3. Varying initial region for benchmarks. Initial set region is $X_0 : (x_1 \in [0, b], \ldots, x_{20} \in [0, b], x_{21} = 0, \ldots, x_n = 0)$ for varying values b. n is the dimension of a original system, k is the dimension of a reduced-order model, $|\delta_s|$ is the total statistical error bound, OOT represents "out of time" with the time limit = 600 s, $T(S_n)$: flowpipe computation time for original model, $T(S_k)$: flowpipe computation on reduced order model, T_M: time taken for model order reduction, $T(\delta_s)$ time taken for error estimation and calculation of k.

Benchmark	b	$T(S_n)$	k	$\lvert\delta_s\rvert$	$T(S_k)$	T_M	$T(\delta_s)$
Analog circuits.1 ($n = 100$) $t_f = 3$	0.0025	59	2	0.06	6	1	5
	0.01	63	2	0.24	7	2	5
	0.05	65	3	1.13	23	3	5
	0.1	67	15	2.1	OOT	1.5	5
Analog circuits.1 ($n = 500$) $t_f = 3$	0.0025	OOT	2	0.13	6	2.2	46
	0.01	OOT	2	0.5	7	1.5	60
	0.05	OOT	3	2.5	24	2	45
	0.1	OOT	16	5	OOT	2.7	45
Analog circuits.2 ($n = 100$) $t_f = 3$	0.0025	136	2	0.06	0.5	3	39
	0.01	140	2	0.24	0.5	3	46
	0.05	140	5	1.1	23	3	39
	0.1	142	16	2.1	OOT	3.6	42
Analog circuits.2 ($n = 500$) $t_f = 3$	0.0025	OOT	2	0.13	0.5	6.4	374
	0.01	OOT	2	0.5	0.6	7	375
	0.05	OOT	5	2.5	23	9	434
	0.1	OOT	17	5	OOT	7	418
Fluid dynamics ($n = 100$) $t_f = 2$	0.0025	66	3	0.074	1.4	3	18
	0.01	73	3	0.22	1.5	3	18
	0.05	72	5	0.98	14.8	3	18
	0.1	83	18	2.0	OOT	3	18
Fluid dynamics ($n = 500$) $t_f = 2$	0.0025	OOT	3	2.3	1.6	9	271
	0.01	OOT	3	2.5	1.5	8.7	294
	0.05	OOT	10	2.5	OOT	8.1	272
	0.1	OOT	17	4.99	OOT	10	277

the time limit of 600 s for these benchmarks. The loss of effectiveness of MOR when the initial set is large, while not unexpected, is a limiting factor for its application to verification problems. Table 4 shows the results. Here, we note that the sensitivity to changes in initial conditions and time horizon is highly model dependent. Nevertheless, we notice a clear trend that as the time horizon is increased, the reduced order models are generally larger and harder to compute with.

Table 4. Varying time horizon t_f for the benchmarks with a fixed parameter $b = 0.01$ of the initial condition region. See Table 3 for the legend.

Benchmark	t_f	$T(S_n)$	k	$\lvert\delta_s\rvert$	$T(S_k)$	T_M	$T(\delta_s)$
Analog circuits.1 ($n = 100$)	1	20	2	0.24	2	0.5	1.5
	3	63	2	0.24	7	1.7	5
	6	213	2	0.24	15	3	9
	12	263	2	0.24	33	5	19
	20	416	3	0.24	160	8.5	27
Analog circuits.1 ($n = 500$)	1	OOT	2	0.52	2	1.9	27
	3	OOT	2	0.52	7	1.5	60
	6	OOT	2	0.52	15	3.1	94
	12	OOT	2	0.52	33	4.3	165
	20	OOT	3	0.52	100	6.3	308
Analog circuits.2 ($n = 100$)	1	47	2	0.24	0.3	1	12
	3	140	2	0.24	0.5	3	46
	6	268	2	0.24	1	5.8	90
	12	502	2	0.24	2	13	175
	20	OOT	3	0.24	13	45	350
Analog circuits.2 ($n = 500$)	1	OOT	2	0.5	0.3	2.5	131
	3	OOT	2	0.5	0.6	7	376
	6	OOT	2	0.52	1	15	OOT
	12	OOT	2	0.52	2	26	OOT
	20	OOT	3	0.52	13	47	OOT
Fluid dynamics ($n = 100$)	1	30	2	0.22	0.3	1.5	18
	2	73	3	0.22	1.5	3	18
	4	158	5	0.24	34	4.4	70
	8	325	8	0.23	OOT	5	100
Fluid dynamics ($n = 500$)	1	OOT	3	0.91	0.6	4.7	143
	2	OOT	3	2.5	1.5	8.7	249
	4	OOT	5	0.59	34	16.8	563
	8	OOT	8	0.52	OOT	35	OOT

5 Conclusion

In conclusion, we have studied the effectiveness of model order reduction methods for nonlinear systems. We note that the approach is promising whenever the singular values show a rapid decay, failing which it does not produce significant reductions. We also note that the reduced order models provide for highly effective verification and are accurate enough for many tasks. Thus, MOR seems effective in a class of systems obtained by discretization of PDEs provided we

are interested in a small set of system variables. Furthermore, MOR becomes more challenging as the initial conditions and time horizon of interest become larger.

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