Intersection Type Calculi of Bounded Dimension

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Abstract

A notion of *dimension* in intersection typed λ-calculi is presented. The dimension of a typed λ-term is given by the minimal *norm* of an *elaboration* (a proof theoretic decoration) necessary for typing the term at its type, and, intuitively, measures intersection introduction as a resource.

Bounded-dimensional intersection type calculi are shown to enjoy subject reduction, since terms can be elaborated in non-increasing norm under β-reduction. We prove that a multiset interpretation (corresponding to a non-idempotent and non-linear interpretation of intersection) of dimensionality corresponds to the number of simultaneous constraints required during search for inhabitants. As a consequence, the inhabitation problem is decidable in bounded multiset dimension, and it is proven to be EXPSPACE-complete. This result is a substantial generalization of inhabitation for the rank 2-fragment, yielding a calculus with decidable inhabitation which is independent of rank.

Our results give rise to a new criterion (dimensional bound) for subclasses of intersection type calculi with a decidable inhabitation problem, which is orthogonal to previously known criteria, and which should have immediate applications in synthesis. Additionally, we give examples of dimensional analysis of fragments of the intersection type system, including conservativity over simple types, rank 2-types, and normal form typings, and we provide some observations towards dimensional analysis of other systems. It is suggested (for future work) that our notion of dimension may have semantic interpretations in terms of of reduction complexity.

Categories and Subject Descriptors F.4.1 [*Mathematical Logic and Formal Languages*]: Mathematical Logic

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1. Introduction

Background. Intersection type systems [2, 10] enjoy a prominent position within the theory of typed λ-calculus [3]. As is well known, variants of such systems characterize deep semantic properties of λ-terms, including normalization and solvability properties [3]. Several applications of intersection type systems in programming language theory exist and are generally related to the capability of intersection types to specify behavioral properties of programs (application areas include refinement types, abstract interpretation, model checking, object calculi, process types, and synthesis, see [20] for an overview).

As a consequence of the enormous expressive power of intersection types, standard type theoretic decision problems [3] are undecidable for general intersection type systems, including the problem of type checking (given a term and a type, does the term have the type?) and inhabitation (given a type, does there exist a term having the type?). It is therefore a long-standing topic of interest to investigate fragments of the system admitting of algorithmic solutions to important decision problems. Some restrictions have been proposed which lead to decidable type checking or type inference, for example [8, 21]. As in the cases of application specific changes to the system mentioned above, such approaches typically work non-uniformly, by changing the rules of the system in different ways.

The present work was motivated out of a desire to find uniform principles of bounding the *inhabitation problem* for λ-calculus with intersection types: Given a type environment Γ and a type σ, does there exist a λ-term M such that $\Gamma \vdash M : \sigma$ is derivable in the intersection type system [2, 10]? This problem is known to be undecidable, as was shown by Urzyczyn [32]. The main known result concerning decidable fragments and fine structure of the problem is given in Urzyczyn's relatively recent subsequent analysis in [33] (improving on [23]), where it is shown that the problem is decidable and EXPSPACE-complete in rank 2, and undecidable in all ranks k for $k \geq 3$ (here referring to Leivant's notion of rank [24]). Quite recently, in [29, 30], the inhabitation problem has been shown to be equivalent to the problem of λ-definability [26], for which Loader proved undecidability in 1993 (but first published in 2001) [25].

Contribution. In this paper we introduce a concept of *dimensionality* for intersection type systems, and we apply it to obtain a new type theoretic principle of bounding of the inhabitation problem: Given Γ, σ and $n > 0$, is there a term M such that $\Gamma \vdash_n M : \sigma$? Here, the relation $\vdash_n$ denotes typability under dimensional bound n.

We develop the notion of dimension from a proof theoretic analysis, both in the standard set theoretic system of intersection types and in a multiset system obtained by a non-idempotent and non-linear interpretation of intersection. In each case, the dimension of a term M at type σ and type environment Γ is a proof theoretic measure of the complexity of intersection needed to type the term at σ. It is witnessed by an *elaboration* (a proof theoretic decoration) of M with minimal *norm* at Γ and σ, which, intuitively, measures intersection introduction as a logical resource.

Bounded-dimensional intersection type calculi are shown to enjoy subject reduction, since terms can be elaborated in non-increasing norm under β-reduction. It turns out that bounded-dimensional inhabitation in set theoretic dimension is undecidable. But we can prove that the multiset interpretation of dimensionality corresponds to the number of simultaneous constraints required during search for inhabitants, characterizing the exponential space model ("bus machines") of [33]. As a consequence, the inhabitation problem is decidable in bounded multiset dimension, and it is

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proven to be EXPSPACE-complete. This result is shown to strictly subsume inhabitation for the rank 2-fragment, thereby yielding a substantially generalized calculus with decidable inhabitation which is independent of rank.

We give examples of dimensional analysis of some well-known fragments of the intersection type system, including conservativity over simple types, rank 2-types, and normal form typings, as well as observations towards dimensional analysis of other systems. These properties together with subject reduction in non-increasing norm suggest (for future work) that our notion of dimension ought to have interpretations in terms of operational semantics, as a proof theoretic (probably coarse-grained) measure of reduction complexity.

While we believe the results presented here to be of independent theoretical interest, they should have immediate applications to program synthesis based on inhabitation and Wajsberg/Ben-Yelles style term enumeration [19] in typed λ-calculi, which is currently being pursued in several lines of work, including [15, 17] which use intersection types (see these references for recent overviews of type-based synthesis). More specific comparisons to directly related technical results in type theory are given in Section 2.

Outline. We outline the main technical development and organization of the paper. We introduce (Section 3) an elaborated version of the strict intersection type system [36], where a judgement $\Gamma \vdash M \longmapsto \mathbf{P} : \sigma$ means that the pure term M has type σ by the elaboration $\mathbf{P}$. Elaborations $\mathbf{P}$ are type-annotated versions of pure terms, in which each subterm occurrence is decorated with the set of types assigned to it in the type derivation. In this set theoretic system, intersection is treated in the standard way as an associative, commutative, idempotent operator. We then (Section 3.2) introduce a corresponding multiset elaboration system, $\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{s}$, where intersections are treated as multisets, appearing in multiset elaborations $\mathbb{P}$ and multiset types $\mathfrak{s}$, corresponding to a non-idempotent interpretation of intersection. One has the property $(\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{s}) \Rightarrow (\Delta^\circ \vdash M \longmapsto \mathbb{P}^\circ : \mathfrak{s}^\circ)$, where $(\_)^\circ$ maps multisets to the underlying sets. The basic idea is now to equip elaborations with a norm, $\|\bullet\|$, such that we can measure the usage of intersection introduction on the elaboration, where $\|\mathbf{P}\|$ and $\|\mathbb{P}\|$ denote the maximal size of decorations in the elaborations. We can then define the dimension of a term M at Γ and σ as the smallest number $n > 0$ such that M has an elaboration at Γ and σ with norm n. We can consider both set theoretic dimension and multiset dimension, depending on which elaboration system we use to measure it, writing $\Gamma \vdash_n M : \sigma$ and $\Gamma \Vdash_n M : \sigma$ to denote typability in dimension n, referring to the set theoretic and the multiset dimension, respectively. More precisely, we define $\Gamma \Vdash_n M : \sigma$ if and only if we have

$$\exists \Delta, \mathbb{P}, \mathfrak{s}. \ (\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{s} \text{ with } \Gamma = \Delta^\circ \text{ and } \sigma = \mathfrak{s}^\circ \text{ and } \|\mathbb{P}\| \leq n)$$

The inhabitation problem in bounded (multiset) dimension can be defined as the decision problem: Given Γ, σ and $n > 0$, does there exist a term λ-term M such that $\Gamma \Vdash_n M : \sigma$?

We develop two fundamental results for bounded-dimensional intersection types. First (Section 4), we prove a quantitative version of subject reduction (Theorem 18), showing preservation of elaboration in bounded dimension under β-reduction:

$$\Gamma \Vdash_n M : \sigma, \ M \twoheadrightarrow_\beta M' \ \Rightarrow \ \Gamma \Vdash_n M' : \sigma$$

(here shown for multiset dimension, analogous property is true in set theoretic dimension). This result is important for understanding inhabitation in bounded dimension, since it allows inhabitant search to be restricted to normal forms. Section 5 contains dimensional analysis of subsystems, including conservativity over simple types, rank 2 types, and normal forms, and some observations on other systems, including System F. In particular, we prove (Proposition 23) that rank 2-typings are typable in linear dimension, which is used later to establish the complexity of inhabitation in bounded multiset dimension, and it implies that inhabitation in bounded dimension strictly subsumes rank 2 inhabitation [33], generalizing across all ranks.

The second main result, developed in Section 6, is soundness and completeness (Theorem 31), with respect to inhabitation in bounded multiset dimension, of inhabitant search by a bounded tree-width alternating search procedure operating on multisets of simultaneous inhabitation constraints. As a consequence, we can prove that inhabitation in bounded multiset dimension is EXPSPACE-complete (Theorem 34), via a correspondence to bus machines [33] with linear bounded tape. This result is in contrast to bounded-dimensional inhabitation in set theoretic dimension, which is shown to be undecidable (Theorem 28). Finally, we record the result (Proposition 35) that inhabitation in linear, non-idempotent types [5] is NP-complete.

Section 7 concludes, and Section 8 has remarks on future work.

For space reasons some proof details have been left out. They can be found in [16].

2. Related Work

We make critical use of Urzyczyn's [33] already mentioned rank-based analysis of inhabitation in terms of systems of simultaneous inhabitation constraints and their connection to the exponential space-bounded computational model of bus machines. Bus machines were further used in [28] to show EXPSPACE-completeness of inhabitation in the fragment of the intersection type system without intersection introduction studied in [22], a system which is in many ways orthogonal to the dimension-bounded systems introduced here and probably less useful in practice. We give some more technical details of comparison in Section 5. Bounding principles applied to combinatory logic in [14, 27] concern depth of types (order, rank) and are fundamentally different from dimensional bound.

In studying quantitative aspects of type derivations based on a multiset interpretation of intersection our approach is broadly related to recent work on implicit computational complexity using principles of (light or soft) linear logic combined with *non-idempotent intersection types*, e.g., [4, 11–13]. However, our focus on inhabitation complexity is quite different from the technical goals pursued in implicit complexity. Interestingly, the notion of depth of intersection introduction in [11] is somewhat related to dimension as studied by us, yet it is different in detail (focusing on depth rather than width) and serves different purposes. The most directly related previous work on non-idempotent systems is the work of Ronchi Della Rocca et al. [5] on the inhabitation problem for non-idempotent intersection types, which was shown to be decidable. A major difference is that our notion of resource is *non-linear*. Whereas the non-idempotent systems mentioned above treat intersection types *linearly* (in the sense of linear logic), we treat intersection *introduction* as a resource, but we do not treat intersection *types* linearly with respect to the usage of term variables. This circumstance causes the systems to have quite different properties. Thus, our notion of multiset dimension is independent of the size of λ-terms, and, for example, all Church numerals have dimension 1 in our system, whereas these terms require ever growing linear non-idempotent types (see our discussion in Section 5 for more details). Logical linearity causes the size of minimally sized inhabitants to be polynomially (in fact, linearly) bounded by type size, and we can show (Proposition 35, Section 6.3) that inhabitation in the system of [5] is NP-complete (the complexity of the problem was not considered in [5]), which is in contrast to EXPSPACE-completeness of inhabitation in bounded dimension studied here. Further investigation of relations between our notion of dimension and the aforementioned linear non-idempotent systems and reduction complexity is an interesting topic for future work.

3. Elaboration Systems

We introduce elaboration systems, extending the intersection type system with λ-terms decorated with sets or multisets of types. Such decorated terms are called elaborations. The goal is to define a norm $\|\bullet\|$ to measure the usage of intersection introduction by measuring the sizes of the decorating sets or multisets on elaborations. We first consider a set theoretic system (Section 3.1), corresponding to the standard interpretation of intersection as an associative, commutative, idempotent operator. We then introduce an analogous multiset system (Section 3.2) which corresponds to a non-idempotent interpretation of intersection. In both cases we define the notion of norm and use it to define our notions of dimension. Our presentation of the intersection type system is based on a stratification of intersection types, into strict types and intersections of such, which is standard [34–36]. Using this representation we can measure the number of components of an intersection uniquely.

3.1 Set Theoretic System

Untyped λ-terms are ranged over by M, N, P, Q, etc.:

$$M, N \quad ::= \quad x \mid (\lambda x.M) \mid (MN)$$

Unless otherwise stated we follow notational conventions of [1]. Following the presentation of intersection types in the so-called strict system of [36, Definition 5.1], referred to here as λ^S, we stratify intersection types into strict types $(A, B, \ldots)$ and strict intersection types $(\sigma, \tau, \ldots)$:

$$\begin{array}{lllll} \mathcal{T}_s & \ni & A, B & ::= & a \mid \sigma \to A \\ \mathcal{T}_{si} & \ni & \sigma, \tau & ::= & A_1 \cap \cdots \cap A_n \end{array}$$

where $a, b, \ldots$ range over atoms, and $n > 0$.[1]

Set theoretic *elaborations* are λ-terms decorated with nonempty finite sets, ranged over by S, of strict types, $S = \{A_1, \ldots, A_n\}$:

$$\mathbf{P}, \mathbf{Q} \quad ::= \quad x[S] \mid (\lambda x.\mathbf{P})[S] \mid (\mathbf{PQ})[S]$$

We let $|S|$ denote the size of the set S. We write decorations $[\{A_1, \ldots, A_n\}]$ in simplified notation as $[A_1, \ldots, A_n]$.

Let $\lceil \mathbf{P} \rceil$ denote the untyped term arising from erasing all decorations from $\mathbf{P}$.

Define the operation $\mathbf{P} \sqcup \mathbf{Q}$ on elaborations $\mathbf{P}, \mathbf{Q}$ with $\lceil \mathbf{P} \rceil \equiv \lceil \mathbf{Q} \rceil$:

$$\begin{array}{lll} x[S] \sqcup x[S'] & \equiv & x[S \cup S'] \\ (\lambda x.\mathbf{P})[S] \sqcup (\lambda x.\mathbf{Q})[S'] & \equiv & (\lambda x.\mathbf{P} \sqcup \mathbf{Q})[S \cup S'] \\ (\mathbf{PQ})[S] \sqcup (\mathbf{P'Q'})[S'] & \equiv & ((\mathbf{P} \sqcup \mathbf{P'})(\mathbf{Q} \sqcup \mathbf{Q'}))[S \cup S'] \end{array}$$

The set of elaborations of an untyped term M is naturally equipped with a partial order, denoted $\sqsubseteq$, defined as the least partial order satisfying:

$$\begin{array}{lll} x[S] \sqsubseteq x[S'] & \Leftrightarrow & S \subseteq S' \\ (\lambda x.\mathbf{P})[S] \sqsubseteq (\lambda x.\mathbf{Q})[S'] & \Leftrightarrow & S \subseteq S' \text{ and } \mathbf{P} \sqsubseteq \mathbf{Q} \\ (\mathbf{PQ})[S] \sqsubseteq (\mathbf{P'Q'})[S'] & \Leftrightarrow & S \subseteq S' \text{ and } \mathbf{P} \sqsubseteq \mathbf{P'} \text{ and } \mathbf{Q} \sqsubseteq \mathbf{Q'} \end{array}$$

In order to relate elaborations to type derivations we introduce an elaborated version of the strict intersection type system [36, Definition 5.1], denoted λ^S and with derivability relation denoted $\vdash_S$. A judgement $\Gamma \vdash M \longmapsto \mathbf{P} : \sigma$ of the elaborated system signifies that the term M elaborates to $\mathbf{P}$ in the environment Γ at σ. Such an elaboration $\mathbf{P}$ of M indicates how the intersection introduction rule $(\cap\mathrm{I})$ has been applied to subterm occurrences of M in order to obtain a typing of M in the strict intersection type system λ^S. We refer to the elaborated system as $\lambda^{[\cap]}$, which is given by the following rules. Notice that the operation $\bigsqcup_{i=1}^n \mathbf{P}_i$ in the conclusion of rule $(\cap\mathrm{I})$ may cause sets in the decorations of elaborations to grow according to the usage of this rule.

$$\frac{}{\Gamma, x : \bigcap_{i=1}^n A_i \vdash x \longmapsto x[A_i] : A_i}(\mathrm{var})$$

$$\frac{\Gamma, x : \sigma \vdash M \longmapsto \mathbf{P} : A}{\Gamma \vdash \lambda x.M \longmapsto (\lambda x.\mathbf{P})[\sigma \to A] : \sigma \to A}(\to\mathrm{I})$$

$$\frac{\Gamma \vdash M \longmapsto \mathbf{P} : \sigma \to A \quad \Gamma \vdash N \longmapsto \mathbf{Q} : \sigma}{\Gamma \vdash (MN) \longmapsto (\mathbf{PQ})[A] : A}(\to\mathrm{E})$$

$$\frac{\Gamma \vdash M \longmapsto \mathbf{P}_i : A_i \ (i = 1 \ldots n)}{\Gamma \vdash M \longmapsto \bigsqcup_{i=1}^n \mathbf{P}_i : \bigcap_{i=1}^n A_i}(\cap\mathrm{I})$$

It is immediate from the definition of λ^S [36, Definition 5.1] that we have

$$\Gamma \vdash_S M : \sigma \Leftrightarrow \exists \mathbf{P}.\, \Gamma \vdash M \longmapsto \mathbf{P} : \sigma$$

and this can be taken here as a definition of λ^S and $\vdash_S$.

Define the *max-norm* $\|\bullet\|$ on elaborations:

$$\begin{array}{lll} \|x[S]\| & = & |S| \\ \|(\lambda x.\mathbf{P})[S]\| & = & \max\{\|\mathbf{P}\|, |S|\} \\ \|(\mathbf{PQ})[S]\| & = & \max\{\|\mathbf{P}\|, \|\mathbf{Q}\|, |S|\} \end{array}$$

Lemma 1. *For all elaborations* $\mathbf{P}$ *and* $\mathbf{Q}$ *with* $\lceil \mathbf{P} \rceil \equiv \lceil \mathbf{Q} \rceil$ *we have*

1. $\|\mathbf{P}\| > 0$
2. $\|\mathbf{P} \sqcup \mathbf{Q}\| \leq \|\mathbf{P}\| + \|\mathbf{Q}\|$
3. $\mathbf{P} \sqsubseteq \mathbf{Q} \Rightarrow \|\mathbf{P}\| \leq \|\mathbf{Q}\|$

Definition 2 ($\lambda_n^{[\cap]}$). *For* $n > 0$ *define* $\vdash_n$*, the bounded-dimensional relation with dimension n, by setting* $\Gamma \vdash_n M : \sigma$ *if and only if*

$$\exists \mathbf{P}.\, \Gamma \vdash M \longmapsto \mathbf{P} : \sigma \text{ with } \|\mathbf{P}\| \leq n$$

We refer to the bounded-dimensional system with dimension n as $\lambda_n^{[\cap]}$.

Clearly, we have

Lemma 3. $\Gamma \vdash_S M : \sigma$ *if and only if* $\Gamma \vdash_n M : \sigma$ *for some* $n > 0$.

Definition 4 (Set theoretic dimension). *We define the* set theoretic dimension *of a term M in* $\lambda^{[\cap]}$ *at type* σ *in environment* Γ *by*

$$\dim_\Gamma^\sigma(M) = \min\{n \mid \Gamma \vdash_n M : \sigma\}$$

and set $\dim_\Gamma^\sigma(M) = \infty$ *if it is not the case that* $\Gamma \vdash_S M : \sigma$*. We write* $\dim^\sigma$ *for* $\dim_\emptyset^\sigma$.

We consider a few examples of elaborations (notation: we sometimes write a set decoration $[A, B, \ldots]$ as a column vector).

Example 5. $I \equiv \lambda x.x$ *elaborates at* $\sigma \equiv (a \to a) \cap (b \to b)$ *to*

$$\vdash \lambda x.x \longmapsto (\lambda x.x[a, b]) \begin{bmatrix} a \to a \\ b \to b \end{bmatrix} : \sigma$$

The elaboration arises as

$$(\lambda x.x[a, b]) \begin{bmatrix} a \to a \\ b \to b \end{bmatrix} \equiv (\lambda x.x[a])[a \to a] \ \sqcup \ (\lambda x.x[b])[b \to b]$$

showing that $\dim^\sigma(I) = 2$.

Let $\mathbf{c}_2 \equiv \lambda f.\lambda x.(f(f\ x))$ *for which we have the elaboration*

$$\begin{array}{l} \vdash \lambda f.\lambda x.(f(f\ x)) \longmapsto \\ \quad (\lambda f.(\lambda x.(f[A]\ (f[A]\ x[a])[a])[a])[A])[B] : B \end{array}$$

for $A \equiv a \to a$, $B = (a \to a) \to a \to a$*, showing that* $\dim^B(\mathbf{c}_2) = 1$.

Let $\omega \equiv \lambda x.(xx)$ *for which we have the elaboration*

$$\vdash \lambda x.(xx) \longmapsto (\lambda x.(x[a \to a]\ x[a])[a])[A] : A$$

for $A \equiv ((a \to a) \cap a) \to a$*, showing that* $\dim^A(\omega) = 1$.

[1] Allowing $n = 0$ would lead to a system comprising of a universal type (the empty intersection), ω. Doing so would be both possible and algebraically interesting, but we leave it out here for simplicity.

A slightly more complicated example, which will be useful in comparing with the multiset system:

Example 6. *Let* $\tau \equiv (a \to a) \cap (b \to b)$ *and* $\sigma \equiv (b \to b) \cap (c \to c)$ *and consider the typing of the term* $M \equiv x(y(\lambda z.z))$ *in the environment* $\Gamma = \{x : (r \cap s) \to e, y : (\tau \to r) \cap (\sigma \to s)\}$. *We have* $\Gamma \vdash M \longmapsto \mathbf{P} : e$, *where*

$$\mathbf{P} \equiv (x[(r \cap s) \to e](y\begin{bmatrix} \tau \to r \\ \sigma \to s \end{bmatrix}(\lambda z.z\begin{bmatrix} a \\ b \\ c \end{bmatrix})\begin{bmatrix} a \to a \\ b \to b \\ c \to c \end{bmatrix})\begin{bmatrix} r \\ s \end{bmatrix})[e]$$

which shows that $\dim^e_\Gamma(M) = 3$.

3.2 Multiset System

Strict multiset types are ranged over by φ, ψ, etc. and multiset types are ranged over by $\mathbb{s}, \mathbb{t}$, etc., which are finite multisets of strict multiset types, $\mathbb{s} : \mathcal{T}_{sm} \to \mathbb{N}_0$:

$$\begin{array}{lcll} \mathcal{T}_{sm} & \ni & \varphi, \psi & ::= \; a \mid \mathbb{s} \to \varphi \\ \mathcal{T}_{m} & \ni & \mathbb{s}, \mathbb{t} & ::= \; \langle \varphi_1, \ldots, \varphi_n \rangle \end{array}$$

where $a, b, \ldots$ range over atoms, and $n > 0$.

Multiset elaborations are λ-terms decorated with nonempty finite multisets $\mathbb{s}$:

$$\mathbb{P}, \mathbb{Q} \quad ::= \quad x\langle \mathbb{s} \rangle \mid (\lambda x.\mathbb{P})\langle \mathbb{s} \rangle \mid (\mathbb{P}\mathbb{Q})\langle \mathbb{s} \rangle$$

We write $\mathbb{s} = \langle \varphi_1, \ldots, \varphi_n \rangle$ where a member φ_i appears $\mathbb{s}(\varphi) > 0$ times in the unordered list. We write a decoration as $\langle \varphi_1, \ldots, \varphi_n \rangle$, as shorthand for $\langle \mathbb{s} \rangle$.

We let $|\mathbb{s}|$ denote the size of the multiset, $|\mathbb{s}| = \sum\{\mathbb{s}(\varphi) \mid \mathbb{s}(\varphi) > 0\}$. We define the operation $\mathbb{s} \vee \mathbb{s}'$ by setting $(\mathbb{s} \vee \mathbb{s}')(\varphi) = \max\{\mathbb{s}(\varphi), \mathbb{s}'(\varphi)\}$, and we define multiset union $\mathbb{s} \uplus \mathbb{s}'$ as usual by $(\mathbb{s} \uplus \mathbb{s}')(\varphi) = \mathbb{s}(\varphi) + \mathbb{s}'(\varphi)$. Multiset containment is denoted $\Subset$, with $\mathbb{s} \Subset \mathbb{s}'$ if and only if $\mathbb{s}(\varphi) \leq \mathbb{s}'(\varphi)$ for all φ.

Let $\lceil \mathbb{P} \rceil$ denote the untyped term arising from erasing all decorations from $\mathbb{P}$. Define the operation $\mathbb{P} \uplus \mathbb{Q}$ on multiset elaborations $\mathbb{P}, \mathbb{Q}$ with $\lceil \mathbb{P} \rceil \equiv \lceil \mathbb{Q} \rceil$:

$$\begin{array}{lcl} x\langle \mathbb{s} \rangle \uplus x\langle \mathbb{s}' \rangle & \equiv & x\langle \mathbb{s} \uplus \mathbb{s}' \rangle \\ (\lambda x.\mathbb{P})\langle \mathbb{s} \rangle \uplus (\lambda x.\mathbb{Q})\langle \mathbb{s}' \rangle & \equiv & (\lambda x.\mathbb{P} \uplus \mathbb{Q})\langle \mathbb{s} \uplus \mathbb{s}' \rangle \\ (\mathbb{P}\mathbb{Q})\langle \mathbb{s} \rangle \uplus (\mathbb{P}'\mathbb{Q}')\langle \mathbb{s}' \rangle & \equiv & ((\mathbb{P} \uplus \mathbb{P}')(\mathbb{Q} \uplus \mathbb{Q}'))\langle \mathbb{s} \uplus \mathbb{s}' \rangle \end{array}$$

The set of multiset elaborations of an untyped term M is naturally equipped with a partial order, denoted $\trianglelefteq$, defined as the least partial order satisfying:

$$\begin{array}{lcl} x\langle \mathbb{s} \rangle \trianglelefteq x\langle \mathbb{s}' \rangle & \Leftrightarrow & \mathbb{s} \Subset \mathbb{s}' \\ (\lambda x.\mathbb{P})\langle \mathbb{s} \rangle \trianglelefteq (\lambda x.\mathbb{Q})\langle \mathbb{s}' \rangle & \Leftrightarrow & \mathbb{s} \Subset \mathbb{s}' \text{ and } \mathbb{P} \trianglelefteq \mathbb{Q} \\ (\mathbb{P}\mathbb{Q})\langle \mathbb{s} \rangle \trianglelefteq (\mathbb{P}'\mathbb{Q}')\langle \mathbb{s}' \rangle & \Leftrightarrow & \mathbb{s} \Subset \mathbb{s}' \text{ and } \mathbb{P} \trianglelefteq \mathbb{P}' \text{ and } \mathbb{Q} \trianglelefteq \mathbb{Q}' \end{array}$$

We introduce a multiset-elaborated version of the strict intersection type system $(\lambda^S, \vdash_S)$ in which a judgement $\Delta \vdash M \Longmapsto \mathbb{P} : \mathbb{s}$ signifies that the term M elaborates to the multiset elaboration $\mathbb{P}$ in the environment Δ at $\mathbb{s}$.

The rules of the multiset elaboration type system, referred as $\lambda^{\langle \cap \rangle}$, are as follows.

$$\frac{}{\Delta, x : \langle \varphi_1, \ldots, \varphi_n \rangle \vdash x \Longmapsto x\langle \varphi_i \rangle : \langle \varphi_i \rangle}(\text{var})$$

$$\frac{\Delta, x : \mathbb{s} \vdash M \Longmapsto \mathbb{P} : \langle \varphi \rangle}{\Delta \vdash \lambda x.M \Longmapsto (\lambda x.\mathbb{P})\langle \mathbb{s} \to \varphi \rangle : \langle \mathbb{s} \to \varphi \rangle}(\to\text{I})$$

$$\frac{\Delta \vdash M \Longmapsto \mathbb{P} : \langle \mathbb{s} \to \varphi \rangle \quad \Delta \vdash N \Longmapsto \mathbb{Q} : \mathbb{s}}{\Delta \vdash (MN) \Longmapsto (\mathbb{P}\mathbb{Q})\langle \varphi \rangle : \langle \varphi \rangle}(\to\text{E})$$

$$\frac{\Delta \vdash M \Longmapsto \mathbb{P}_i : \langle \varphi_i \rangle \; (i = 1 \ldots n) \quad (\star)}{\Delta \vdash M \Longmapsto \biguplus_{i=1}^n \mathbb{P}_i : \langle \varphi_1, \ldots, \varphi_n \rangle}(\cap\text{I})$$

where $(\star)$ is the side condition for all free variables x in M:

$$\bigvee_x (\biguplus_{i=1}^n \mathbb{P}_i) \Subset \Delta(x)$$

Here the expression $\bigvee_x(\mathbb{P})$ is defined by

$$\bigvee_x(\mathbb{P}) = \bigvee\{\mathbb{s} \mid x\langle \mathbb{s} \rangle \text{ occurs as a subexpression in } \mathbb{P}\}$$

Informally, the side condition ensures that the environment Δ assumes at least the maximal intersection type resources that are used in *distinct occurrences* of variables x in the elaboration. Notice that this condition is different from tracking resources linearly (compare Section 2 and Section 5), and that our management of type environments is non-linear. As already indicated (Section 1 and Section 2), we thereby treat intersection *introduction* as a resource, but not intersection *types* as such.

We define translations between strict intersection types and multiset types as follows. We assume below that in types of the form $A_1 \cap \cdots \cap A_n$ we have $A_i \neq A_j$ for $i \neq j$.

$$\begin{array}{lcl} a^* & \equiv & a \\ (\sigma \to A)^* & \equiv & \langle \sigma^* \rangle \to A^*, \text{ if } \sigma = B \text{ for some } B \\ (\sigma \to A)^* & \equiv & \sigma^* \to A^*, \text{ otherwise} \\ (A_1 \cap \cdots \cap A_n)^* & \equiv & \langle A_1{}^*, \ldots, A_n{}^* \rangle \end{array}$$

$$\begin{array}{lcl} a^\circ & \equiv & a \\ (\mathbb{s} \to \varphi)^\circ & \equiv & \mathbb{s}^\circ \to \varphi^\circ \\ \langle \varphi_1, \ldots, \varphi_n \rangle^\circ & \equiv & \varphi_1{}^\circ \cap \cdots \cap \varphi_n{}^\circ \end{array}$$

The translations are lifted pointwise to type environments and to elaborations in the obvious way.

Clearly, we have

$$(\Delta \vdash M \Longmapsto \mathbb{P} : \mathbb{s}) \;\Rightarrow\; (\Delta^\circ \vdash M \longmapsto \mathbb{P}^\circ : \mathbb{s}^\circ)$$

and

$$\begin{array}{lcl} (\Gamma \vdash M \longmapsto \mathbf{P} : \sigma) & \Rightarrow & \exists \Delta, \mathbb{s}, \mathbb{P}.\; \Delta \vdash M \Longmapsto \mathbb{P} : \mathbb{s} \\ & & \quad \text{where } \Delta^\circ = \Gamma, \mathbb{s}^\circ = \sigma, \mathbb{P}^\circ = \mathbf{P} \end{array}$$

Define the *max-norm* $||\bullet||$ on multiset elaborations:

$$\begin{array}{lcl} ||x\langle \mathbb{s} \rangle|| & = & |\mathbb{s}| \\ ||(\lambda x.\mathbb{P})\langle \mathbb{s} \rangle|| & = & \max\{||\mathbb{P}||, |\mathbb{s}|\} \\ ||(\mathbb{P}\mathbb{Q})\langle \mathbb{s} \rangle|| & = & \max\{||\mathbb{P}||, ||\mathbb{Q}||, |\mathbb{s}|\} \end{array}$$

Lemma 7. *For multiset elaborations* $\mathbb{P}$, $\mathbb{Q}$ *with* $\lceil \mathbb{P} \rceil \equiv \lceil \mathbb{Q} \rceil$ *we have*

1. $||\mathbb{P}|| > 0$
2. $||\mathbb{P} \uplus \mathbb{Q}|| \leq ||\mathbb{P}|| + ||\mathbb{Q}||$
3. $\mathbb{P} \trianglelefteq \mathbb{Q} \Rightarrow ||\mathbb{P}|| \leq ||\mathbb{Q}||$

We could now define a bounded-dimensional version of the multiset system in direct analogy with Definition 2 and Definition 4, using the condition

$$\exists \mathbb{P}.\; \Delta \vdash M \Longmapsto \mathbb{P} : \mathbb{s} \text{ with } ||\mathbb{P}|| \leq n$$

While doing so may indeed be interesting, in this paper we wish to use the multiset system to characterize proof complexity in the standard set theoretic intersection type system. We therefore focus on the following concept of dimensional bound, which relates multiset elaborations to the set theoretic system and allows us to ascribe multiset dimension to that system.

Definition 8 ($\lambda_n^{\langle \cap \rangle}$). *For* $n > 0$ *define* $\Vdash_n$, *the bounded multiset-dimensional relation with dimension n, by setting* $\Gamma \Vdash_n M : \sigma$ *if and only if*

$$\exists \Delta, \mathbb{P}, \mathbb{s}.\; (\Delta \vdash M \Longmapsto \mathbb{P} : \mathbb{s} \textit{ with } \Gamma = \Delta^\circ \textit{ and } \sigma = \mathbb{s}^\circ \textit{ and } ||\mathbb{P}|| \leq n)$$

We refer to the bounded-dimensional system with dimension n as $\lambda_n^{\langle \cap \rangle}$.

Clearly, we have

Lemma 9. $\Gamma \vdash_S M : \sigma$ *if and only if* $\Gamma \Vdash_n M : \sigma$ *for some* $n > 0$.

Definition 10 (Multiset dimension). *We define the* multiset dimension *of a term M in $\lambda^{\langle\cap\rangle}$ at type σ in environment Γ by*

$$\mathrm{Dim}^{\sigma}_{\Gamma}(M) = \min\{n \mid \Gamma \Vdash_n M : \sigma\}$$

and set $\mathrm{Dim}^{\sigma}_{\Gamma}(M) = \infty$ *if it is not the case that there exists n such that $\Gamma \Vdash_n M : \sigma$. We write* Dim^{σ} *for* $\mathrm{Dim}^{\sigma}_{\emptyset}$ *and* $\mathrm{Dim}^{\mathfrak{s}}_{\Delta}$ *for* $\mathrm{Dim}^{\mathfrak{s}^{\circ}}_{\Delta^{\circ}}$.

Example 11. *Consider* $\mathbf{c}_2 \equiv \lambda f.\lambda x.(f(f\ x))$ *for which we have the multiset elaboration*

$$\begin{array}{l} \vdash \lambda f.\lambda x.(f(f\ x)) \longmapsto \\ \quad (\lambda f.(\lambda x.(f\langle\varphi\rangle\ (f\langle\varphi\rangle\ x\langle a\rangle)\langle a\rangle)\langle a\rangle)\langle\varphi\rangle)\langle\psi\rangle : \langle\psi\rangle \end{array}$$

for $\varphi \equiv \langle a\rangle \to a$, $\psi = \langle\langle a\rangle \to a\rangle \to \langle a\rangle \to a$, *showing that* $\mathrm{Dim}^{\langle\psi\rangle}(\mathbf{c}_2) = 1$. *In fact, for every n we have* $\mathrm{Dim}^{\langle\psi\rangle}(\mathbf{c}_n) = 1$. *Even more, every simply typed term evidently has multiset dimension* 1 *at every one of its simple types.*

Consider $\omega \equiv \lambda x.(xx)$ *with the multiset elaboration*

$$\vdash \lambda x.(xx) \longmapsto (\lambda x.(x\langle\langle a\rangle \to a\rangle\ x\langle a\rangle)\langle a\rangle)\langle\varphi\rangle : \langle\varphi\rangle$$

for $\varphi \equiv \langle\langle a\rangle \to a, a\rangle \to a$, *showing that* $\mathrm{Dim}^{\langle\varphi\rangle}(\omega) = 1$.

It is interesting to compare the following example with Example 6. It is also interesting to note (for later) that the typing in this example exceeds rank 2.

Example 12. *Let* $\tau \equiv (a \to a)\cap(b \to b)$, $\sigma \equiv (b \to b)\cap(c \to c)$ *and consider again typing the term* $M \equiv x(y(\lambda z.z))$ *in the environment* $\Gamma = \{x : (r \cap s) \to e, y : (\tau \to r) \cap (\sigma \to s)\}$. *We have*

$$\{x : \langle\left\langle \begin{array}{c} r \\ s \end{array} \right\rangle \to e\rangle, y : \left\langle \begin{array}{c} \tau^* \to r \\ \sigma^* \to s \end{array} \right\rangle\} \vdash (x(y(\lambda z.z))) \Longrightarrow \mathbb{P} : \langle e\rangle$$

where $\mathbb{P}$ *is the multiset elaboration*

$$(x\langle\left\langle \begin{array}{c} r \\ s \end{array} \right\rangle \to e\rangle(y\left\langle \begin{array}{c} \tau^* \to r \\ \sigma^* \to s \end{array} \right\rangle(\lambda z.z\left\langle \begin{array}{c} a \\ b \\ b \\ c \end{array} \right\rangle)\left\langle \begin{array}{c} a \to a \\ b \to b \\ b \to b \\ c \to c \end{array} \right\rangle)\left\langle \begin{array}{c} r \\ s \end{array} \right\rangle)\langle e\rangle$$

which shows that $\mathrm{Dim}^{e}_{\Gamma}(M) = 4$.

4. Subject Reduction

We prove subject reduction in detail for multiset elaborations, because the property is more challenging for this system. The property also holds for the set theoretic system by a similar argument, see [16, Appendix B].

For a λ-term M and variable x with $k \geq 0$ free occurrences in M we write $M_x[x_1, \ldots, x_k]$ to denote the linearization of M with respect to x, that is, the term arising from replacing each j'th free occurrence of x in M with a distinct fresh variable x_j, for $j = 1 \ldots k$ (in case $k = 0$, i.e., x does not occur free in M, we have $M_x[x_1, \ldots, x_k] \equiv M$). We use the shorthand notation $M_x[x_j]^k_{j=1}$ for $M_x[x_1, \ldots, x_k]$. For an elaboration $\mathbb{P}$ with $k \geq 0$ free occurrences of variable x we write $\mathbb{P}_x[\mathbb{Q}_1, \ldots, \mathbb{Q}_k]$ for the elaboration which arises by replacing each j'th subexpression $x\langle\mathfrak{s}_j\rangle$ by $\mathbb{Q}_j$ in $\mathbb{P}$, and we use the shorthand notation $\mathbb{P}_x[\mathbb{Q}_j]^k_{j=1}$. Notice that an elaboration of the form $\mathbb{P}_x[\mathbb{Q}_j]^k_{j=1}$ can be regarded as the result of filling each j'th hole with $\mathbb{Q}_j$, for $j = 1 \ldots k$, in a multi-hole context containing k holes in place of the subexpressions $x\langle\mathfrak{s}_j\rangle$. We will sometimes use the shorthand notation $\{x_j : \mathfrak{s}_j\}^k_{j=1}$ for $\bigcup^k_{j=1}\{x_j : \mathfrak{s}_j\}$.

We outline the main ideas in the following proof of subject reduction under bounded dimension. We consider a redex $R \equiv (\lambda x.M)N$ with elaboration

$$\mathbb{R} \equiv ((\lambda x.\mathbb{P})\langle\mathfrak{s} \to \varphi\rangle\ \mathbb{Q})\langle\varphi\rangle$$

such that

$$\Delta, x : \mathfrak{s} \vdash M \Longrightarrow \mathbb{P} : \langle\varphi\rangle$$

and

$$\Delta \vdash N \Longrightarrow \mathbb{Q} : \mathfrak{s}$$

The basic idea is now to analyze elaboration of a substitution $M\{x := N\}$ by means of the linearization $M_x[x_j]^k_{j=1}$ of M with respect to x. We can show that an elaboration $\mathbb{Q}$ of N may be "decomposed" into k smaller elaborations $\mathbb{Q}_j$ such that we have $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$ for $j = 1 \ldots k$, and such that $M\{x := N\}$ elaborates to $\mathbb{R}' \equiv \mathbb{P}_x[\mathbb{Q}_j]^k_{j=1}$, and it can be shown that $\|\mathbb{R}'\| \leq \|\mathbb{R}\|$. The side condition $(\star)$ $\bigvee_x(\biguplus^n_{i=1} \mathbb{P}_i) \mathrel{\underline{\subsetplus}} \Delta(x)$ in rule $(\cap\mathrm{I})$ is used, in the proof of Lemma 14, to establish that $M_x[x_j]^k_{j=1}$ can be elaborated under type assumptions $x_j : \mathfrak{s}_j$ such that $\bigvee^k_{j=1} \mathfrak{s}_j \mathrel{\underline{\subsetplus}} \mathfrak{s}$. This latter condition is used, in turn, to show substitutivity (Lemma 16) by appealing to decomposition (Lemma 15): If $\Delta \vdash N \Longrightarrow \mathbb{Q} : \mathfrak{s}$ and $\bigvee^k_{j=1} \mathfrak{s}_j \mathrel{\underline{\subsetplus}} \mathfrak{s}$ (implying $\mathfrak{s}_j \mathrel{\underline{\subsetplus}} \mathfrak{s}$ for all $j = 1 \ldots k$), then using Lemma 15 we can show $\Delta \vdash N \Longrightarrow \mathbb{Q}_j : \mathfrak{s}_j$ for some $\mathbb{Q}_j$ with $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$. This property is exploited to organize a somewhat delicate inductive proof of substitutivity (Lemma 16) which constitutes the main part of the subject reduction proof. A basic problem solved there using the properties above is that elaboration under bounded norm is not a priori preserved inductively at intersection introduction: From inductive premises $\Delta \vdash M\{x := N\} \Longrightarrow \mathbb{P}_i : \langle\varphi_i\rangle$ with $\|\mathbb{P}_i\| \leq \|\mathbb{R}\|$ we cannot conclude that $\|\biguplus^n_{i=1} \mathbb{P}_i\| \leq \|\mathbb{R}\|$.

Lemma 13 (Distributivity). *For any elaborations* $\mathbb{P}^i$, $\mathbb{Q}^i_j$ $(i = 1 \ldots n$, $j = 1 \ldots k)$ *such that* $\lceil\mathbb{P}^p\rceil \equiv \lceil\mathbb{P}^q\rceil$ *and* $\lceil\mathbb{Q}^p_j\rceil \equiv \lceil\mathbb{Q}^q_j\rceil$ *for all* $p = 1 \ldots n, q = 1 \ldots n$, $j = 1 \ldots k$, *and such that x has k free occurrences in the* $\mathbb{P}^i$, *one has*

$$\biguplus^{n}_{i=1} \mathbb{P}^i_x[\mathbb{Q}^i_j]^k_{j=1} \equiv (\biguplus^{n}_{i=1} \mathbb{P}^i)_x[\biguplus^{n}_{i=1} \mathbb{Q}^i_j]^k_{j=1}$$

Proof: For $k = 1$ prove $\biguplus_i \mathbb{P}^i_x[\mathbb{Q}_i] \equiv (\biguplus_i \mathbb{P}^i)_x[\biguplus_i \mathbb{Q}^i]$ by induction on $\lceil\mathbb{P}^1\rceil$. Then proceed by induction on k. □

Lemma 14 (Linearization). *Suppose M has k occurrences of the free variable x, denoted $x_{(j)}$ for $j = 1 \ldots k$, and assume we have* $\Delta, x : \mathfrak{s} \vdash M \Longrightarrow \mathbb{P} : \mathfrak{t}$, *where the subexpressions* $x_{(j)}\langle\mathfrak{s}_j\rangle$ *occur in* $\mathbb{P}$ *for $j = 1 \ldots k$. Then one has*

$$\Delta \cup \{x_j : \mathfrak{s}_j\}^k_{j=1} \vdash M_x[x_j]^k_{j=1} \Longrightarrow \mathbb{P}_x[x_j\langle\mathfrak{s}_j\rangle]^k_{j=1} : \mathfrak{t}$$

with $\bigvee^k_{j=1} \mathfrak{s}_j \mathrel{\underline{\subsetplus}} \mathfrak{s}$, *where the x_j are fresh.*

Proof: The proof is by induction on the derivation of $\Delta, x : \mathfrak{s} \vdash M \Longrightarrow \mathbb{P} : \mathfrak{t}$. In the case of rule $(\cap I)$ the side condition $(\star)$ is used to establish the condition $\bigvee^k_{j=1} \mathfrak{s}_j \mathrel{\underline{\subsetplus}} \mathfrak{s}$. More details can be found in [16, Appendix A]. □

The following lemma is a "quantitative" version of the well-known fact (see [36, Section 5]) that the rule $(\cap\mathrm{E})$ of intersection elimination (in fact, more generally, the subtyping rule) is admissible in the strict intersection type system.

Lemma 15 (Decomposition). *Assume* $\Delta \vdash M \Longrightarrow \mathbb{Q} : \mathfrak{s}$ *with* $\mathfrak{s} = \mathfrak{s}_1 \uplus \mathfrak{s}_2$. *Then there exist $\mathbb{Q}_1$ and $\mathbb{Q}_2$ such that the following conditions hold:*

1. $\Delta \vdash M \Longrightarrow \mathbb{Q}_1 : \mathfrak{s}_1$
2. $\Delta \vdash M \Longrightarrow \mathbb{Q}_2 : \mathfrak{s}_2$
3. $\mathbb{Q}_1 \uplus \mathbb{Q}_2 \equiv \mathbb{Q}$
4. $\|\mathbb{Q}\| \geq |\mathfrak{s}| = |\mathfrak{s}_1| + |\mathfrak{s}_2|$

Proof: Immediate, by inversion of rule $(\cap\mathrm{I})$. □

LEMMA 16 (Substitutivity). *Suppose that*

$$\Delta \cup \{x_j : \mathfrak{s}_j\}_{j=1}^k \vdash M_x[x_j]_{j=1}^k \Longmapsto \mathbb{P}_x[x_j\langle\mathfrak{s}_j\rangle]_{j=1}^k : \mathfrak{t}$$

and $\Delta \vdash N \Longmapsto \mathbb{Q} : \mathfrak{s}$ *with* $\bigvee_{j=1}^k \mathfrak{s}_j \mathrel{\underline{\Subset}} \mathfrak{s}$. *Then there exist* $\mathbb{Q}_1, \dots, \mathbb{Q}_k$ *such that* $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$ *for* $j = 1 \dots k$ *and*

$$\Delta \vdash M_x[x_j]_{j=1}^k\{x_j := N\}_{j=1}^k \Longmapsto \mathbb{P}_x[\mathbb{Q}_j]_{j=1}^k : \mathfrak{t}$$

Proof: By induction on the derivation of the judgement

$$\Delta \cup \{x_j : \mathfrak{s}_j\}_{j=1}^k \vdash M_x[x_j]_{j=1}^k \Longmapsto \mathbb{P}_x[x_j\langle\mathfrak{s}_j\rangle]_{j=1}^k : \mathfrak{t}$$

and we proceed by cases over the last rule used. We show only the case where $(\cap\mathrm{I})$ is used last, the remaining cases of the proof can be found in [16, Appendix A]. So consider an application of $(\cap\mathrm{I})$ with premises

$$\Delta \cup \{x_j : \mathfrak{s}_j\}_{j=1}^k \vdash M_x[x_j]_{j=1}^k \Longmapsto \mathbb{P}_x^i[x_j\langle\mathfrak{s}_j^i\rangle]_{j=1}^k : \langle\varphi_i\rangle \tag{1}$$

for $i = 1 \dots n$, with $\mathfrak{t} = \langle\varphi_1, \dots, \varphi_n\rangle$ and

$$\biguplus_{i=1}^n \mathbb{P}_x^i[x_j\langle\mathfrak{s}_j^i\rangle]_{j=1}^k \equiv \mathbb{P}_x[x_j\langle\mathfrak{s}_j\rangle]_{j=1}^k \tag{2}$$

It follows from linearity of the x_j in $M_x[x_j]_{j=1}^k$ and $\mathbb{P}_x^i[x_j\langle\mathfrak{s}_j^i\rangle]_{j=1}^k$ that the assumptions $x_j : \mathfrak{s}_j^i$ suffice, so we can strengthen (1) to:

$$\Delta \cup \{x_j : \mathfrak{s}_j^i\}_{j=1}^k \vdash M_x[x_j]_{j=1}^k \Longmapsto \mathbb{P}_x^i[x_j\langle\mathfrak{s}_j^i\rangle]_{j=1}^k : \langle\varphi_i\rangle \tag{3}$$

for $i = 1 \dots n$. Lemma 13 implies

$$\biguplus_{i=1}^n \mathbb{P}_x^i[x_j\langle\mathfrak{s}_j^i\rangle]_{j=1}^k \equiv (\biguplus_{i=1}^n \mathbb{P}^i)_x[x_j\langle\biguplus_{i=1}^n \mathfrak{s}_j^i\rangle]_{j=1}^k$$

which together with (2) implies that we have

$$\mathfrak{s}_j = \biguplus_{i=1}^n \mathfrak{s}_j^i \text{ for } j = 1 \dots k \tag{4}$$

By $\Delta \vdash N \Longmapsto \mathbb{Q} : \mathfrak{s}$ and the assumption $\bigvee_{j=1}^k \mathfrak{s}_j \mathrel{\underline{\Subset}} \mathfrak{s}$ we get from Lemma 15 together with (4) that we have, for some $\mathbb{Q}_1^i, \dots, \mathbb{Q}_k^i$ $(i = 1 \dots n)$

$$\Delta \vdash N \Longmapsto \mathbb{Q}_j^i : \mathfrak{s}_j^i \text{ with } \biguplus_{i=1}^n \mathbb{Q}_j^i \trianglelefteq \mathbb{Q} \text{ for} j = 1 \dots k \tag{5}$$

We now apply the induction hypothesis to (3) and (5), which shows that there exist $\mathbb{R}_j^i$ $(j = 1 \dots k, i = 1 \dots n)$ such that

$$\Delta \vdash M_x[x_j]_{j=1}^k\{x_j := N\}_{j=1}^k \Longmapsto \mathbb{P}_x^i[\mathbb{R}_j^i]_{j=1}^k : \langle\varphi_i\rangle \tag{6}$$

with $\mathbb{R}_j^i \trianglelefteq \mathbb{Q}_j^i$ $(j = 1 \dots k, i = 1 \dots n)$. It follows that we have

$$\biguplus_{i=1}^n \mathbb{R}_j^i \trianglelefteq \biguplus_{i=1}^n \mathbb{Q}_j^i$$

and hence by (5)

$$\biguplus_{i=1}^n \mathbb{R}_j^i \trianglelefteq \mathbb{Q} \text{ for } j = 1 \dots k \tag{7}$$

We apply rule $(\cap\mathrm{I})$ to (6) and obtain

$$\Delta \vdash M_x[x_j]_{j=1}^k\{x_j := N\}_{j=1}^k \Longmapsto \biguplus_{i=1}^n \mathbb{P}_x^i[\mathbb{R}_j^i]_{j=1}^k : \mathfrak{t} \tag{8}$$

Now, by Lemma 13 and (2) we have

$$\biguplus_{i=1}^n \mathbb{P}_x^i[\mathbb{R}_j^i]_{j=1}^k \equiv (\biguplus_{i=1}^n \mathbb{P}^i)_x[\biguplus_{i=1}^n \mathbb{R}_j^i]_{j=1}^k \equiv \mathbb{P}_x[\biguplus_{i=1}^n \mathbb{R}_j^i]_{j=1}^k \tag{9}$$

Take $\mathbb{Q}_j \equiv \biguplus_{i=1}^n \mathbb{R}_j^i$ for $j = 1 \dots k$. Then $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$ is true for $j = 1 \dots k$, by (7), and the lemma is proven by (8) and (9). ▫

LEMMA 17 (Substitution under non-increasing norm). *Assume that* $\Delta, x : \mathfrak{s} \vdash M \Longmapsto \mathbb{P} : \mathfrak{t}$ *and* $\Delta \vdash N \Longmapsto \mathbb{Q} : \mathfrak{s}$. *Then there exists an elaboration* $\mathbb{R}$ *such that*

$$\Delta \vdash M\{x := N\} \Longmapsto \mathbb{R} : \mathfrak{t}$$

with $\|\mathbb{R}\| \leq \max\{\|\mathbb{P}\|, \|\mathbb{Q}\|\}$.

Proof: Assume that M has $k \geq 0$ free occurrences of x. By Lemma 14 we have

$$\Delta \cup \{x_j : \mathfrak{s}_j\}_{j=1}^k \vdash M_x[x_j]_{j=1}^k \Longmapsto \mathbb{P}_x[x_j\langle\mathfrak{s}_j\rangle]_{j=1}^k : \mathfrak{t}$$

with $\bigvee_{j=1}^k \{\mathfrak{s}_j\} \mathrel{\underline{\Subset}} \mathfrak{s}$, where the x_j are fresh. By Lemma 16 there exist $\mathbb{Q}_1, \dots, \mathbb{Q}_k$ such that $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$ for $j = 1 \dots k$ and

$$\Delta \vdash M_x[x_j]_{j=1}^k\{x_j := N\}_{j=1}^k \Longmapsto \mathbb{P}_x[\mathbb{Q}_j]_{j=1}^k : \mathfrak{t}$$

Clearly, we have

$$M_x[x_j]_{j=1}^k\{x_j := N\}_{j=1}^k \equiv M\{x := N\}$$

and

$$\|\mathbb{P}_x[\mathbb{Q}_j]_{j=1}^k\| = \max\{\|\mathbb{P}\|, \max_{j=1}^k\{\|\mathbb{Q}_j\|\}\}$$

Since $\mathbb{Q}_j \trianglelefteq \mathbb{Q}$ for $j = 1 \dots k$, we have $\max_{j=1}^k\{\|\mathbb{Q}_j\|\} \leq \|\mathbb{Q}\|$ and hence

$$\max\{\|\mathbb{P}\|, \max_{j=1}^k\{\|\mathbb{Q}_j\|\}\} \leq \max\{\|\mathbb{P}\|, \|\mathbb{Q}\|\}$$

which proves the lemma taking $\mathbb{R} \equiv \mathbb{P}_x[\mathbb{Q}_j]_{j=1}^k$. ▫

THEOREM 18 (Subject Reduction in bounded dimension). *Assume* $\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{t}$ *and* $M \twoheadrightarrow_\beta M'$. *Then there exists an elaboration* $\mathbb{R}$ *such that* $\Delta \vdash M' \Longmapsto \mathbb{R} : \mathfrak{t}$ *and* $\|\mathbb{R}\| \leq \|\mathbb{P}\|$. *In particular,* $\Gamma \Vdash_n M : \tau$ *implies* $\Gamma \Vdash_n M' : \tau$.

Proof: Consider an elaboration $\mathbb{P}$ of a redex $(\lambda x.P')N$ with $\mathbb{P} \equiv ((\lambda x.\mathbb{P}')\langle\mathfrak{s} \to \varphi\rangle\ \mathbb{Q})\langle\varphi\rangle$. We have $\|\mathbb{P}\| = \max\{\|\mathbb{P}'\|, \|\mathbb{Q}\|\}$. By Lemma 17, we have an elaboration $\mathbb{R}$ of $P'\{x := N\}$ with $\|\mathbb{R}\| \leq \max\{\|\mathbb{P}'\|, \|\mathbb{Q}\|\}$. The details on generalization to reduction in context are given in [16, Appendix A]. ▫

Similarly, subject reduction holds for the set theoretic system, see [16, Appendix B].

THEOREM 19 (Subject Reduction in bounded dimension). *Assume* $\Gamma \vdash M \longmapsto \mathbf{P} : \tau$ *and* $M \twoheadrightarrow_\beta M'$. *Then there exists an elaboration* $\mathbf{R}$ *such that* $\Gamma \vdash M' \longmapsto \mathbf{R} : \tau$ *and* $\|\mathbf{R}\| \leq \|\mathbf{P}\|$. *In particular,* $\Gamma \vdash_n M : \tau$ *implies* $\Gamma \vdash_n M' : \tau$.

5. Dimensional Analysis

We give a few results that show how dimensionality can be used to characterize various subsystems of the intersection type system. We focus on low-complexity systems, namely simple types (Proposition 20), rank 2 typings (Proposition 23), and normal form typings (Proposition 25). We briefly compare with linear, non-idempotent systems and indicate how dimension can be compared across systems with different type languages, using System F as an example.

More specifically, Proposition 20 and Proposition 23 show that dimensionality can be indicative of logical complexity (subsystems characterized by lower dimensionality). The result on rank 2-typings (Proposition 23) is central. It shows that the rank 2-fragment is a special case of linear bounded dimensionality, and it will furthermore be important later for establishing the complexity of bounded-dimensional inhabitation (Theorem 34). The dimensional characterization of normal forms (Proposition 25) indicates that dimension may be an interesting semantic measure related to the complexity of β-reduction for λ-terms. Using an "absolute" notion of dimension (quantifying out dependency on types) we can compare across systems, exemplified by System F (Example 26). Finally, a technical tool in our analysis is the notion of *leaf-norm* giving a handle on multiset norm which will also be useful for establishing complexity of bounded-dimensional inhabitation later.

It is easy to see (as illustrated in Example 11) that the simple typed λ-calculus lives entirely within multiset dimension 1. It is less obvious what happens if we consider inhabitants of simple types in the intersection type system. But, using a classical result on intersection types ([2, Corollary 4.10]), we immediately have:

Proposition 20 (Dimension over simple types). *Let Γ be an environment consisting of simple types, let σ be a simple type, and let N be a normal form. Then $\Gamma \vdash_S N : \sigma$ implies $\Gamma \Vdash_1 N : \sigma$.*

Proof: Suppose $\Gamma \vdash_S N : \sigma$ with Γ and σ as stated. By the Conservativity property [2, Corollary 4.10], $\Gamma \vdash N : \sigma$ is derivable in the simple type system of Curry. Such a typing can obviously be elaborated at multiset dimension 1. □

We introduce an alternative characterization of the max-norm over multiset elaborations, which will be very useful throughout the remainder of this paper. We define the *leaf-norm* $\|\bullet\|_L$ on multiset elaborations:

$$\begin{array}{lcl} \|x\langle\mathfrak{s}\rangle\|_L & = & |\mathfrak{s}| \\ \|(\lambda x.\mathbb{P})\langle\mathfrak{s}\rangle\|_L & = & \|\mathbb{P}\|_L \\ \|(\mathbb{P}\ \mathbb{Q})\langle\mathfrak{s}\rangle\|_L & = & \max\{\|\mathbb{P}\|_L, \|\mathbb{Q}\|_L\} \end{array}$$

An elaboration $\mathbb{P}$ is called *well-typed* if $\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{s}$ for some Δ, M and $\mathfrak{s}$.

Lemma 21. *Let $\mathbb{P}$ be a well-typed multiset elaboration. If either $\mathbb{P} \equiv (\lambda x.\mathbb{Q})\langle\varphi_1, \ldots, \varphi_n\rangle$ or $\mathbb{P} \equiv (\mathbb{Q}\ \mathbb{R})\langle\varphi_1, \ldots, \varphi_n\rangle$, then $\|\mathbb{Q}\| \geq n$.*

Proof: If $n = 1$, the claim is trivially true. Otherwise, by well-typedness of $\mathbb{P}$, it must be the case that $\mathbb{P}$ is typed by $(\cap\mathrm{I})$ so that either $\mathbb{P} \equiv \biguplus_{i=1}^n (\lambda x.\mathbb{Q}_i)\langle\varphi_i\rangle$ or $\mathbb{P} \equiv \biguplus_{i=1}^n (\mathbb{Q}_i\ \mathbb{R}_i)\langle\varphi_i\rangle$ for some $\mathbb{Q}_i, \mathbb{R}_i$ with $\biguplus_{i=1}^n \mathbb{Q}_i = \mathbb{Q}$. Since for each subterm $x\langle\mathfrak{s}\rangle$ in $\mathbb{Q}_i$ for $i = 1 \ldots n$ one has $\|x\langle\mathfrak{s}\rangle\| \geq 1$, we have $\|\mathbb{Q}\| = \|\biguplus_{i=1}^n \mathbb{Q}_i\| \geq n$. □

Lemma 22 (Norm equivalence). *For any well-typed multiset elaboration $\mathbb{P}$ one has $\|\mathbb{P}\| = \|\mathbb{P}\|_L$.*

Proof: By induction on $\lceil\mathbb{P}\rceil$.

In case $\mathbb{P} \equiv x\langle\mathfrak{s}\rangle$, we have $\|\mathbb{P}\| = |\mathfrak{s}| = \|\mathbb{P}\|_L$.

In case $\mathbb{P} \equiv (\lambda x.\mathbb{Q})\langle\mathfrak{s}\rangle$, Lemma 21 shows $\|\mathbb{P}\| = \max\{\|\mathbb{Q}\|, |\mathfrak{s}|\} = \|\mathbb{Q}\|$. By definition of $\|\bullet\|_L$ and the induction hypothesis we have $\|\mathbb{P}\|_L = \|(\lambda x.\mathbb{Q})\langle\mathfrak{s}\rangle\|_L = \|\mathbb{Q}\|_L = \|\mathbb{Q}\|$, which shows the claim.

In case $\mathbb{P} \equiv (\mathbb{Q}\ \mathbb{R})\langle\mathfrak{s}\rangle$, Lemma 21 again shows that $\|\mathbb{P}\| = \max\{\|\mathbb{Q}\|, \|\mathbb{R}\|, |\mathfrak{s}|\} = \max\{\|\mathbb{Q}\|, \|\mathbb{R}\|\}$. By the definition of $\|\bullet\|_L$ and the induction hypothesis, we have $\|\mathbb{P}\|_L = \|(\mathbb{Q}\ \mathbb{R})\langle\mathfrak{s}\rangle\|_L = \max\{\|\mathbb{Q}\|_L, \|\mathbb{R}\|_L\} = \max\{\|\mathbb{Q}\|, \|\mathbb{R}\|\}$, which shows the claim. □

The analogous property does not hold for set theoretic elaborations. Consider the elaborations (only some decorations shown):

$$\begin{array}{lcl} \mathbf{P}_1 & \equiv & (\lambda x.\lambda y.y[b])[a \to b \to b] \\ \mathbf{P}_2 & \equiv & (\lambda x.\lambda y.y[b])[c \to b \to b] \\ \mathbf{P}_1 \sqcup \mathbf{P}_2 & \equiv & (\lambda x.\lambda y.y[b])[a \to b \to b, c \to b \to b] \end{array}$$

We have $\|\mathbf{P}_1 \sqcup \mathbf{P}_2\| = 2$, but $\|\mathbf{P}_1 \sqcup \mathbf{P}_2\|_L = 1$ (assuming the leaf norm $\|\bullet\|_L$ defined on set theoretic elaborations analogously to the definition above). Hence, the max-norm $\|\bullet\|$ remains indispensable for comparing the two elaboration systems.

The following proposition shows that rank 2 elaborations of normal forms have dimension linear in the number of components in the intersection type. It implies that inhabitation in rank 2 interesection types [33] is subsumed by bounded-dimensional inhabitation as developed in Section 6 below, and it will be important in our proof of the complexity of inhabitation (see Theorem 34).

Following Leivant [24] we define the rank, r, of an intersection type by setting $r(\sigma) = 0$ when σ is a simple type, and otherwise $r(\sigma \to A) = \max\{r(\sigma) + 1, r(A)\}$, $r(A_1 \cap \cdots \cap A_n) = \max\{1, r(A_1), \ldots, r(A_n)\}$. The definition is transferred to multiset types in the obvious way. The rank of a typing judgement $\Gamma \vdash M : \sigma$ is $r(\sigma)$, if all types in Γ are simple and otherwise $\max\{r(\Gamma)+1, r(\sigma)\}$ where $r(\Gamma)$ is the maximal rank of a type in Γ. This notion is extended to elaboration judgements in the obvious way.

Proposition 23 (Dimension in rank 2). *Suppose we can derive the judgement $\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi_1, \ldots, \varphi_n\rangle$ in rank 2 where N is a normal form, then $\|\mathbb{P}\| = n$. In particular, if $\Gamma \vdash_S N : \bigcap_{i=1}^n A_i$ in rank 2, then $\Gamma \Vdash_n N : \bigcap_{i=1}^n A_i$, and if $\Gamma \vdash_S N : A$ in rank 2, then $\Gamma \Vdash_1 N : A$.*

Proof: We first prove the implication

$$(\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi_1, \ldots, \varphi_n\rangle) \Rightarrow (\|\mathbb{P}\| = n)$$

by induction on the derivation of $\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi_1, \ldots, \varphi_n\rangle$.

In case rule (var) is used, the claim is clear.

In case rule $(\to\mathrm{I})$ is the last rule used, as

$$\frac{\Delta, x : \mathfrak{s} \vdash N' \Longmapsto \mathbb{Q} : \langle\varphi\rangle}{\Delta \vdash \lambda x.N' \Longmapsto (\lambda x.\mathbb{Q})\langle\mathfrak{s} \to \varphi\rangle : \langle\mathfrak{s} \to \varphi\rangle}(\to\mathrm{I})$$

induction hypothesis shows that $\|\mathbb{Q}\| = 1$, and the claim evidently follows.

In case rule $(\cap\mathrm{I})$ is the last rule used, as

$$\frac{\Delta \vdash M \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle\ (i = 1 \ldots n) \quad (\star)}{\Delta \vdash M \Longmapsto \biguplus_{i=1}^n \mathbb{P}_i : \langle\varphi_1, \ldots, \varphi_n\rangle}(\cap\mathrm{I})$$

induction hypothesis implies $\|\mathbb{P}_i\| = 1$ for $i = 1 \ldots n$, and we therefore have, by Lemma 7, $\|\biguplus_{i=1}^n \mathbb{P}_i\| \leq \sum_{i=1}^n \|\mathbb{P}_i\| \leq n$. On the other hand, since $\|\mathbb{P}_i\| \geq 1$, we have $\|\mathbb{P}_i\|_L \geq 1$, and therefore evidently $\|\biguplus_{i=1}^n \mathbb{P}_i\|_L \geq n$, so (Lemma 22) $\|\biguplus_{i=1}^n \mathbb{P}_i\| \geq n$ which proves the claim in this case.

In case rule $(\to\mathrm{E})$ is the last rule used, we can assume the following situation, by rank 2 restriction and normal form:

$$\Delta \vdash x \Longmapsto x\langle\langle\psi_1\rangle \to \cdots \to \langle\psi_k\rangle \to \varphi\rangle : \langle\langle\psi_1\rangle \to \cdots \to \langle\psi_k\rangle \to \varphi\rangle$$

where by rank 2 restriction the ψ_j are all simple types for $j = 1 \ldots k$, and with

$$\Delta \vdash N_j \Longmapsto \mathbb{Q}_j : \langle\psi_j\rangle$$

for $j = 1 \ldots k$, such that the conclusion elaboration $\mathbb{P}$ is given as

$$\Delta \vdash xN_1 \ldots N_k \Longmapsto ((x\langle\ldots\rangle\ \mathbb{Q}_1)\langle\ldots\rangle \ldots \mathbb{Q}_k)\langle\varphi\rangle : \langle\varphi\rangle$$

leaving out a few annotations for readability. By induction we have $\|\mathbb{Q}_j\| = 1$ for $j = 1 \ldots k$ from which it follows that $\|\mathbb{P}\| = 1$, showing the claim. This concludes the inductive proof.

Now suppose that $\Gamma \vdash_S N : \bigcap_{i=1}^n A_i$ in rank 2. Then we have an elaboration $\Delta \vdash N \Longmapsto \mathbb{Q} : \mathfrak{s}$ in rank 2 with $\Delta^\circ = \Gamma$ and $\mathfrak{s}^\circ = \bigcap_{i=1}^n A_i$ for some $\mathbb{Q}$, Δ and $\mathfrak{s}$. It follows from Lemma 15 that we also have $\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi_1, \ldots, \varphi_n\rangle$ for some $\mathbb{P}$ and φ_i with $\varphi_i{}^\circ = A_i$, for $i = 1 \ldots n$. By the property already shown, we have $\|\mathbb{P}\| = n$. □

Notice that no comparable property can be shown for rank 3 elaborations, since we can use rank 3 types to pump up dimensions arbitrarily by creating "chain reactions" among intersection types at applications of variables, as is illustrated in Example 12.

We can generalize Proposition 23 as follows. A proof can be found in [16, Appendix C]. Let T, U range over types defined as follows:

$$\begin{array}{lcl} T & ::= & a \mid U \to T \\ U & ::= & a \mid (\bigcap_{i=1}^n T_i) \to U \end{array}$$

Notice that types of the form T, U are not rank-bounded.

Proposition 24. *Let N be a normal form and let Γ consist of assumptions of the form $(x : \bigcap_{i=1}^{k} T_i)$. If $\Gamma \vdash_S N : \bigcap_{i=1}^{n} U_i$, then $\Gamma \Vdash_n N : \bigcap_{i=1}^{n} U_i$.*

Next we show that every normal form can be typed at multiset dimension 1. The construction in the proof of Proposition 25 is essentially an elaborated version of the construction of the *principal schemes* for normal forms given in [9, Definition 8], and hence the proposition provides a dimensional analysis of the type derivations for such schemes – they have multiset dimension 1.

Proposition 25 (Dimension of normal forms). *For every normal form N there exist Γ and A such that $\Gamma \Vdash_1 N : A$.*

Proof: We show the following property by induction on normal forms N: For every normal form N there exist Δ, $\mathbb{P}$ and φ such that $\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi\rangle$ and $\|\mathbb{P}\| = 1$. The proposition follows from this property by taking $\Gamma = \Delta^\circ$ and $A = \varphi^\circ$.

To prove the property, consider the case where $N \equiv x$, a variable. Here we can take $\Delta = \{x : \langle\varphi\rangle\}$ for an arbitrary strict type φ, and we have $\Delta \vdash x \Longmapsto x\langle\varphi\rangle : \langle\varphi\rangle$, showing the claim.

In case $N \equiv x\ N_1 \dots N_k$ we have by induction hypothesis $\Delta_j \vdash N_j \Longmapsto \mathbb{P}_j : \langle\psi_j\rangle$ with $\|\mathbb{P}_j\| = 1$ for some Δ_j and ψ_j, for $j = 1 \dots k$. Let $\Delta = (\biguplus_{j=1}^{k} \Delta_j) \uplus \{x : \langle\langle\psi_1\rangle \to \cdots \to \langle\psi_k\rangle \to \varphi\rangle\}$, where $(\Delta_1 \uplus \Delta_2)(x) = \Delta_1(x) \uplus \Delta_2(x)$ when $x \in \mathrm{dom}(\Delta_1) \cap \mathrm{dom}(\Delta_2)$, and the elaboration (leaving out some annotations for readability)

$$\mathbb{P} \equiv (x\langle\langle\psi_1\rangle \to \cdots \to \langle\psi_k\rangle \to \varphi\rangle\ \mathbb{P}_1 \dots \mathbb{P}_k)\langle\varphi\rangle$$

Then we have $\Delta \vdash N \Longmapsto \mathbb{P} : \langle\varphi\rangle$. Moreover, we have $\|\mathbb{P}\| = \max\{1, \|\mathbb{P}_1\|, \dots, \|\mathbb{P}_k\|\} = 1$, thereby showing the claim.

In case $N \equiv \lambda x.N'$, we have by induction hypothesis

$$\Delta, x : \mathfrak{s} \vdash N' \Longmapsto \mathbb{P}' : \langle\psi\rangle$$

for some Δ, $\mathfrak{s}$, ψ and $\mathbb{P}'$ with $\|\mathbb{P}'\| = 1$. Taking $\mathbb{P} \equiv (\lambda x.\mathbb{P}')\langle\mathfrak{s} \to \psi\rangle$ we have $\Delta \vdash \lambda x.N' \Longmapsto \mathbb{P} : \langle\mathfrak{s} \to \psi\rangle$. We can take $\varphi = \mathfrak{s} \to \psi$ and we have $\|\mathbb{P}\| = \|\mathbb{P}'\| = 1$, thereby proving the claim. $\square$

Comparing to type complexity in non-idempotent systems based on linearity [5], dimensionality is independent of term size, whereas terms M with n occurrences of a free variable x require the type of x to be typed with at least n intersection type components in such systems. For example, the Church numerals $\mathbf{c}_n \equiv \lambda f.\lambda x.(f^n x)$ require non-idempotent types $[\alpha \to \alpha]^n \to \alpha \to \alpha$ (where $[\tau]^n$ denotes $\tau \cap \cdots \cap \tau$ with τ appearing n times) of ever growing size, whereas these terms are all typable in multiset dimension 1 (Proposition 20 and Proposition 25). Further comparison to the non-idempotent system of [5] is provided by Proposition 35, which shows that inhabitation in that system is NP-complete. [2]

Considering Theorem 18 together with Proposition 25, the conclusion appears inescapable that norm and dimension must be systematically related to the operational semantics of β-reduction in some way (reduction is non-increasing wrt. norm, and once a normal form is reached, dimension will have dropped to 1 at some type). In general, linear systems such as [5] evidently provide much more fine-grained control over reduction semantics than norm and dimension would provide. Further study of operational interpretations of dimension is needed but must be left to future work.

We can apply our theory of dimension to other systems than subsystems of intersection types. To this end, let us define the notions of *absolute set theoretic dimension* and *absolute multiset dimension*, denoted $\dim^*$ and Dim^* respectively, as

$$\begin{aligned} \dim^*(M) &= \min\{n \mid \exists\Gamma.\exists\sigma.\ \Gamma \vdash_n M : \sigma\} \\ \mathrm{Dim}^*(M) &= \min\{n \mid \exists\Gamma.\exists\sigma.\ \Gamma \Vdash_n M : \sigma\} \end{aligned}$$

Since these notions only depend on the term M, it makes sense to compare typability in absolute dimension across different type systems with possibly different type languages. Because the intersection type system types exactly the set of strongly normalizing terms, this idea can be applied to any system having the strong normalization property. The following example shows a few properties of System F (polymorphic λ-calculus), in particular, System F is seen to type terms of arbitrarily high dimension.

Example 26. *Consider the term ω from Example 5 and Example 11, and define with $\mathbf{I} \equiv \lambda x.x$ the terms $\omega^k(\mathbf{I})$ (for $k \in \mathbb{N}$) by setting $\omega^1(\mathbf{I}) \equiv \omega\ \mathbf{I}$ and $\omega^{k+1}(\mathbf{I}) \equiv \omega(\omega^k(\mathbf{I}))$. It can be seen that the terms $\omega^k(\mathbf{I})$ are typable in System F, and that $\dim^*(\omega^k(\mathbf{I})) = k + 1$ for all $k \in \mathbb{N}$. Moreover, we have $\mathrm{Dim}^*(\omega^k(\mathbf{I})) = 2^k$. In particular, $(\omega\ \mathbf{I})$ is typable at $\langle\langle a\rangle \to a\rangle$ in multiset dimension 2 using the elaboration*

$$\begin{array}{l} (\omega \left\langle \left\langle \begin{array}{c} \langle a\rangle \to a \\ \langle\langle a\rangle \to a\rangle \to \langle a\rangle \to a \end{array} \right\rangle \to \langle a\rangle \to a \right\rangle \\ \mathbf{I} \left\langle \begin{array}{c} \langle a\rangle \to a \\ \langle\langle a\rangle \to a\rangle \to \langle a\rangle \to a \end{array} \right\rangle)\langle\langle a\rangle \to a\rangle \end{array}$$

In order to type $\omega(\omega^k(\mathbf{I}))$, two elaborations of the argument $\omega^k(\mathbf{I})$ are combined, doubling the dimension according to the underlying multiset operations.

Let $\mathbf{K} \equiv \lambda x.\lambda y.x$. The term $(\lambda xy.y\ (x\ \mathbf{I})\ (x\ \mathbf{K}))\ \omega$ is strongly normalizing but is not typable in System F [18, Thm. 13]. It can be seen to be typable at

$$\langle\langle\langle\varphi_1\rangle \to \langle\langle\varphi_2\rangle \to \varphi_2\rangle \to a\rangle \to a\rangle$$

in multiset dimension 2, where $\varphi_1 = \langle a\rangle \to a$, $\varphi_2 = \langle a\rangle \to \langle a\rangle \to a$. Note that the argument x needs to be assigned two different types of incompatible shapes.

6. Bounded-Dimensional Inhabitation

The bounded-dimensional inhabitation problem in set theoretic dimension is the following decision problem INH_{dim}:

- Given an environment Γ, a type σ and a number $n > 0$: does there exist a term M such that $\Gamma \vdash_n M : \sigma$?

The bounded-dimensional inhabitation problem in multiset dimension is the following decision problem INH_{Dim}:

- Given an environment Γ, a type σ and a number $n > 0$: does there exist M such that $\Gamma \Vdash_n M : \sigma$?

Unfolding definitions, this latter problem means:

- Given an environment Γ, a type σ and a number $n > 0$: does there exist M, Δ, $\mathfrak{s}$ and $\mathbb{P}$ such that $\Gamma = \Delta^\circ$, $\sigma = \mathfrak{s}^\circ$, and $\Delta \vdash M \Longmapsto \mathbb{P} : \mathfrak{s}$ with $\|\mathbb{P}\| \leq n$?

We prove that the problem INH_{Dim} is EXPSPACE-complete. The main technical result towards this end is Theorem 31 (Section 6.2) which shows that, in effect (spelled out in Theorem 34, Section 6.3), bounded multiset dimension is a logical correlate of the linear tape bus machines of [33]. Intuitively, multiset dimension corresponds to the maximal number of simultaneous inhabitation constraints that need to be processed in an alternating search procedure for inhabitants.

[2] A few remarks may also be in order to compare with the system without intersection introduction [22]. In this system, in many ways orthogonal to the dimension-bounded systems, intersection logic is effectively limited to subtyping (which we do not consider in this paper), recovering limited forms of intersection introduction via distributivity $\sigma \to (\tau_1 \cap \tau_2) = (\sigma \to \tau_1) \cap (\sigma \to \tau_2)$. Inhabitation is EXPSPACE-complete [28] (and without subtyping PSPACE-complete, in a class with simple types [31]). The system is less useful in practice, since it cannot assign non-trivial intersection types to abstractions. For example, $\lambda x.x$ does not have any types of the form $(A \to A) \cap (B \to B)$, but only $A \cap B \to A \cap B$. In the presence of subtyping we can show that there are typings $\Gamma \vdash M : \sigma$ such that multiset dimension is exponential in the size of Γ and σ.

Before we turn to the problem INH_{Dim}, we show that, perhaps surprisingly at first glance, dimensional bound does not lead to a decision procedure[3] for the set theoretic system: The problem INH_{dim} is undecidable (Theorem 28). Let us recall the following property for the intersection type system.

Lemma 27 (Subformula property, [2] Lemma 4.5). *Let N be a λ-term in normal form. If $\Gamma \vdash N \longmapsto \mathbf{P} : \sigma$, then there exists a derivation of this judgement in which every type is a subformula of σ or of some type appearing in Γ.*

Theorem 28. *The problem INH_{dim} is undecidable.*

Proof: We have $\Gamma \vdash_S M : \sigma$ if and only if $\Gamma \vdash M \longmapsto \mathbf{P} : \sigma$ for some n and some $\mathbf{P}$ with $\|\mathbf{P}\| \leq n$. By strong normalization [10] and subject reduction of $\lambda^{[\cap]}$ (Theorem 19), it is sufficient to consider inhabitation by normal forms. By the subformula property (Lemma 27) it is sufficient to consider inhabitation by normal form terms with typing derivations all of whose formulae are subformulae of the input types in Γ or σ. Because there are at most N distinct subformulae for an input (Γ, σ) of size N, it is sufficient to consider inhabitation in $\lambda_N^{[\cap]}$. The theorem now follows from the undecidability of the inhabitation problem for intersection types [32]. ▫

We describe a decision procedure, called $\mathscr{A}_{\langle d\rangle}$ below, for the problem INH_{Dim} bounded in multiset dimension with parameter d. In Section 6.2 we prove soundness and completeness of $\mathscr{A}_{\langle d\rangle}$. Section 6.3 contains the proof of EXPSPACE-completeness. The lower bound uses bus machines [33] together with Proposition 23. We conclude by recording, for comparison, that inhabitation in linear non-idempotent types [5] is NP-complete.

6.1 Decision Procedure $\mathscr{A}_{\langle d\rangle}$

The decision procedure $\mathscr{A}_{\langle d\rangle}$ will be developed starting from a semi-decision procedure, called $\mathscr{I}_{\langle\bullet\rangle}$, which will then be bounded with a parameter d, resulting in a procedure called $\mathscr{I}_{\langle d\rangle}$. Finally, we will further modify procedure $\mathscr{I}_{\langle d\rangle}$ to obtain procedure $\mathscr{A}_{\langle d\rangle}$.

6.1.1 Procedure $\mathscr{I}_{\langle\bullet\rangle}$

The following Wajsberg/Ben-Yelles style procedure ([6, 33]) is an adaptation of Urzyczyn's semi-decision procedure for inhabitation in the intersection type system [2]. In our formulation we specialize the procedure of Urzyczyn [33] to strict intersection types for the strict intersection type system. Furthermore, we add comments to the procedure, under the heading "*Induced elaboration*", to indicate how runs of the procedure induce elaborations. Finally, we formulate the procedure as processing *multisets* of simultaneous inhabitation constraints (rather than sets of such).

The procedure is alternating [7] and transforms multisets of simultaneous constraints of the form

$$\langle \Gamma_1 \vdash \mathcal{X} : A_1, \ldots, \Gamma_n \vdash \mathcal{X} : A_n \rangle$$

where the type environments $\Gamma_1, \ldots, \Gamma_n$ have the same domain, and $\mathcal{X}$ denotes an unknown inhabitant. Such multisets of constraints are also referred to as *configurations* of procedure $\mathscr{I}_{\langle\bullet\rangle}$ and are ranged over by C. We sometimes write configurations as

$$\langle \Gamma_1 \vdash ? : A_1, \ldots, \Gamma_n \vdash ? : A_n \rangle$$

Whenever such a multiset of simultaneous constraints is processed by the procedure, the procedure searches for a *single* solution to all the constraints, i.e., a normal-form λ-term $\mathcal{X} \equiv N$ such that

$$\Gamma_1 \vdash N : A_1, \ldots, \Gamma_n \vdash N : A_n$$

[3] For the sake of completeness we note here that neither does limiting the arity of the intersection type operator – it is easy to see that the constructions of undecidability in [33] remain effective under such restriction.

The procedure nondeterministically guesses a normal solution $\mathcal{X}$ by repeatedly transforming configurations, choosing one of the following two steps in which parts of the solution are constructed. The process continues until the current constraint system is in a trivially solvable form (step 2 below). At each step, a configuration C is transformed to another configuration C', and the steps of the procedure thereby determine a transformation relation among configurations denoted $C \mapsto C'$. The transformation steps are as follows.

1. If every A_i is a function type, $A_i \equiv \sigma_i \to B_i$, then a possible guess is $\mathcal{X} \equiv \lambda x.\mathcal{Y}$, and provided *(i)* there is no variable y having type σ_i in every Γ_i we transform the constraint system into
$$\langle \Gamma_1, x : \sigma_1 \vdash \mathcal{Y} : B_1, \ldots, \Gamma_n, x : \sigma_n \vdash \mathcal{Y} : B_n \rangle$$
otherwise *(ii)* we can identify the variables x and y and keep the Γ_i unchanged.

   Induced elaboration: Assuming *(i)* that a solution $\mathcal{Y} \equiv N$ to the transformed system elaborates to $\Delta_i, x : \mathfrak{s}_i \vdash N \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle$, for $i = 1 \ldots n$, a solution $\mathcal{X} \equiv \lambda x.N$ elaborates to
$$\Delta_i \vdash \lambda x.N \Longmapsto (\lambda x.\mathbb{P}_i)\langle \mathfrak{s}_i \to \varphi_i\rangle : \langle \mathfrak{s}_i \to \varphi_i\rangle$$
   Assuming *(ii)* that a solution $\mathcal{Y} \equiv N$ to the transformed system elaborates to $\Delta_i \vdash N \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle$, $i = 1 \ldots n$, a solution $\mathcal{X} \equiv \lambda y.N$ elaborates to
$$\Delta_i \vdash \lambda y.N \Longmapsto (\lambda y.\mathbb{P}_i)\langle \mathfrak{s}_i \to \varphi_i\rangle : \langle \mathfrak{s}_i \to \varphi_i\rangle$$

2. If for some variable x and number k we have
$$\Gamma_i \vdash x : \sigma_i^1 \to \cdots \to \sigma_i^k \to A_i$$
for each $i = 1 \ldots n$, then we may guess that $\mathcal{X} \equiv x\mathcal{Y}^1 \cdots \mathcal{Y}^k$ and consider k systems C_j for $j = 1 \ldots k$, where C_j is given by the following configuration, assuming $\sigma_i^j \equiv \bigcap_{q=1}^{m_{ij}} B_q^{ij}$, $i = 1 \ldots n$, $j = 1 \ldots k$,
$$\langle \Gamma_1 \vdash \mathcal{Y}^j : B_1^{1j}, \ldots, \Gamma_1 \vdash \mathcal{Y}^j : B_{m_{1j}}^{1j}, \ldots, \Gamma_n \vdash \mathcal{Y}^j : B_1^{nj}, \ldots, \Gamma_n \vdash \mathcal{Y}^j : B_{m_{nj}}^{nj} \rangle$$
Each of these k systems C_j ($j = 1 \ldots k$) must now be solved independently in parallel (universal transition [7]). If $k = 0$, then $\mathcal{X} \equiv x$ is a solution and the procedure accepts.

   Induced elaboration: If $k = 0$, we have the elaboration
$$\{x : \langle A_i{}^*\rangle\} \vdash x \Longmapsto x\langle A_i{}^*\rangle : \langle A_i{}^*\rangle$$
   Otherwise, assuming that solutions for $\mathcal{Y}^j \equiv N^j$ to the transformed systems elaborate to
$$\Delta_{ij} \vdash N^j \Longmapsto \mathbb{P}_q^{ij} : \langle\psi_q^{ij}\rangle$$
   for $i = 1 \ldots n$, $j = 1 \ldots k$, $q = 1 \ldots m_{ij}$, we have elaborations (eliding some annotations for readability) for $i = 1 \ldots n$:
$$\Delta_i' \vdash xN^1 \cdots N^k \Longmapsto \mathbb{R}_i : \langle A_i{}^*\rangle$$
   where $\mathbb{R}_i \equiv (x\langle\mathfrak{s}_i'\rangle\ \mathbb{Q}^{i1} \cdots \mathbb{Q}^{ik})\langle A_i{}^*\rangle$ with $\mathbb{Q}^{ij} \equiv \biguplus_{q=1}^{m_{ij}} \mathbb{P}_q^{ij}$, $\mathfrak{s}_i' \equiv \mathfrak{s}^{i1} \to \cdots \to \mathfrak{s}^{ik} \to A_i{}^*$, $\mathfrak{s}^{ij} \equiv \biguplus_{q=1}^{m_{ij}} \langle\psi_q^{ij}\rangle$, $\Delta_i' \equiv (\biguplus_{j=1}^{k} \Delta_{ij}) \uplus \{x : \mathfrak{s}_i'\}$, where $(\Delta_1 \uplus \Delta_2)(x) = \Delta_1(x) \uplus \Delta_2(x)$ when $x \in \mathrm{dom}(\Delta_1) \cap \mathrm{dom}(\Delta_2)$, for $i = 1 \ldots n$, $j = 1 \ldots k$.

The *degree* of a configuration C of procedure $\mathscr{I}_{\langle\bullet\rangle}$ is given by its length, i.e., the multiset size of C.

The set of possible runs of procedure $\mathscr{I}_{\langle\bullet\rangle}$ is given by the set of possible configuration transformation sequences of the procedure. The set of runs of the procedure when started from a configuration C determines a set of *computation trees*, denoted $T_C^{\mathscr{I}}$, as follows. The elements of $T_C^{\mathscr{I}}$ are labeled trees with labels drawn from the set of configurations reachable from C. A tree $t \in T_C^{\mathscr{I}}$ is determined by a run as follows. The root of t is labeled C. There is an edge in t from a node labeled C_1 to a node labeled C_2, denoted $C_1 \mapsto_{\lambda x} C_2$, if the run transforms C_1 into C_2 by step 1 adding the variable x to the environments. There are edges in t from a node labeled C_1 to nodes labeled C_2^j for $j = 1 \ldots k$, denoted $C_1 \mapsto^j_{@x} C_2^j$, if the run transforms C_1 into the C_2^j in step 2 by choosing variable x and making the universal transitions to C_2^j for $j = 1 \ldots k$. A computation tree $t \in T_C^{\mathscr{I}}$ is *accepting* if and only if all leaves of t are labeled with accepting configurations according to step 2 of procedure $\mathscr{I}_{\langle\bullet\rangle}$. Clearly, an accepting tree $t \in T_C^{\mathscr{I}}$ is isomorphic to a normal inhabitant of the strict intersection type system.

6.1.2 Procedure $\mathscr{I}_{\langle d\rangle}$

For every number $d > 0$ we obtain procedure $\mathscr{I}_{\langle d\rangle}$ from procedure $\mathscr{I}_{\langle\bullet\rangle}$ by rejecting whenever a configuration C is reached with degree exceeding the parameter d. That is, procedure $\mathscr{I}_{\langle d\rangle}$ rejects at step 1, if $n > d$, and it rejects at step 2 if $\sum_{i=1}^{n} m_{ij} > d$ for any $j = 1 \ldots k$.

The notion of an accepting computation tree can evidently be applied to procedure $\mathscr{I}_{\langle d\rangle}$ as well, by adjusting for the restricted acceptance conditions of that procedure.

6.1.3 Procedure $\mathscr{A}_{\langle d\rangle}$

For every $d > 0$ the decision procedure $\mathscr{A}_{\langle d\rangle}$ is obtained from procedure $\mathscr{I}_{\langle d\rangle}$ by rejecting on repeating configurations along all paths of the computation tree of $\mathscr{I}_{\langle d\rangle}$. Procedure $\mathscr{A}_{\langle d\rangle}$ can evidently be obtained from procedure $\mathscr{I}_{\langle d\rangle}$ by storing the configurations reached along the computation trees of $\mathscr{I}_{\langle d\rangle}$ and rejecting whenever a configuration is reached which has been reached previously along a computation path.

Notation: In the sequel we write $\mathscr{I}_{\langle d\rangle}(C)$ and $\mathscr{A}_{\langle d\rangle}(C)$ to indicate runs of the procedures starting from configuration C.

6.2 Soundness and Completeness

Analogously to the semi-decision procedure of [33] it should be clear that procedure $\mathscr{I}_{\langle\bullet\rangle}$ is sound and complete for the strict intersection type system, in the following sense:

- (Soundness of $\mathscr{I}_{\langle\bullet\rangle}$) Let $C = \langle \Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$. Every accepting computation tree in $T_C^{\mathscr{I}}$ is isomorphic to a normal inhabitant N such that $\Gamma \vdash_S N : \bigcap_{i=1}^n A_n$.
- (Completeness of $\mathscr{I}_{\langle\bullet\rangle}$) Whenever $\Gamma \vdash_S M : \bigcap_{i=1}^n A_i$, there exists a normal inhabitant N such that $\Gamma \vdash_S N : \bigcap_{i=1}^n A_i$ and N is isomorphic to an accepting tree in $T_C^{\mathscr{I}}$ for $C = \langle \Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$.

The corresponding soundness and completeness properties for procedure $\mathscr{I}_{\langle d\rangle}$ with respect to bounded-dimensional inhabitation are the following:

- (Soundness of $\mathscr{I}_{\langle d\rangle}$). If $\mathscr{I}_{\langle d\rangle}(C)$ succeeds with $C = \langle \Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$, then $\Gamma \Vdash_d M : \bigcap_{i=1}^n A_i$ for some M.
- (Completeness of $\mathscr{I}_{\langle d\rangle}$). Whenever $\Gamma \Vdash_d M : \bigcap_{i=1}^n A_i$ for some M, then $\mathscr{I}_{\langle d\rangle}(C)$ succeeds with $C = \langle \Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$.

However, soundness and completeness for procedure $\mathscr{I}_{\langle d\rangle}$ with respect to d-bounded inhabitation does not follow automatically from soundness and completeness of procedure $\mathscr{I}_{\langle\bullet\rangle}$. The obstacle with respect to completeness is that procedure $\mathscr{I}_{\langle\bullet\rangle}$ does not produce all normal forms (it is not an exhaustive inhabitant enumerator or recognizer), due to restrictions needed to ensure termination. First, in step 1, the procedure collapses variables having the same types in all of the Γ_i. Second, in step 2, the procedure does not unfold the computation tree any further in the acceptance case where $k = 0$ (a variable is chosen as inhabitant, which becomes a leaf of the computation tree). As a consequence, even though there might be a tree in $T_C^{\mathscr{I}}$ isomorphic to some normal inhabitant N at Γ and σ, the procedure might conceivably not be able to recognize the existence of any normal form in dimension d, although such an inhabitant exists. We therefore need to establish that, effectively, procedure $\mathscr{I}_{\langle\bullet\rangle}$ always has accepting trees at minimal norm d, for which inhabitants exist, and, furthermore, that such trees are also accepted by procedure $\mathscr{I}_{\langle d\rangle}$. We note that such considerations are not standard in the literature on inhabitation. For example, many inhabitation procedures are regulated to search only for η-long normal forms (see [19]), which makes inhabitant search slightly more deterministic than ours, since choices between step 1 and step 2 become determined by the goal types. However, this does not work in our setting, because dimension is not invariant under η-expansion. Consider $\Gamma = \{f : (a \cap b) \to c\}$ with the goal $\Gamma \vdash ? : (a \cap b) \to c$. One inhabitant is the term f itself, at dimension 1. Another is the η-expanded term $\lambda x.f x$. But this term only elaborates at dimension 2 to $(\lambda x.(f\langle\langle a,b\rangle \to c\rangle\, x\langle a,b\rangle)\langle c\rangle)\langle\langle a,b\rangle \to c\rangle$.

The following is the main lemma for soundness.

LEMMA 29. *Assume procedure $\mathscr{I}_{\langle d\rangle}(C)$ accepts from configuration $C = \langle \Gamma_1 \vdash ? : A_1, \ldots, \Gamma_n \vdash ? : A_n\rangle$ with induced elaborations $\Delta_1 \vdash M \Longrightarrow \mathbb{P}_1 : \langle\varphi_1\rangle, \ldots, \Delta_n \vdash M \Longrightarrow \mathbb{P}_n : \langle\varphi_n\rangle$. Then one has*

1. *$\Delta_i \vdash M \Longrightarrow \mathbb{P}_i : \langle\varphi_i\rangle$ is derivable for $i = 1 \ldots n$*
2. *$\Delta_i^\circ \subseteq \Gamma_i$ and $\varphi_i^\circ = A_i$ for $i = 1 \ldots n$*
3. *$\|\biguplus_{i=1}^n \mathbb{P}_i\|_L \le d$*

Proof: By induction on the depth of the computation tree of an accepting run of $\mathscr{I}_{\langle d\rangle}(C)$.

In case the procedure accepts with $k = 0$ in step 2, the claim is obviously true.

In case the procedure accepts with $k > 0$ in step 2, we have by induction hypothesis solutions $\mathcal{Y}^j \equiv N^j$ with derivable elaborations

$$\Delta_i \vdash N^j \Longrightarrow \mathbb{P}_q^{ij} : \langle\psi_q^{ij}\rangle$$

with $\Delta_i^\circ \subseteq \Gamma_i$ and $(\psi_q^{ij})^\circ = B_q^{ij}$ for $i = 1 \ldots n$, $j = 1 \ldots k$, and such that

$$\|\biguplus_{i=1}^{n} \biguplus_{q=1}^{m_{ij}} \mathbb{P}_q^{ij}\|_L \le d \tag{10}$$

for $j = 1 \ldots k$. It is easy to check that claims 1 and 2 follow for the corresponding elaborations

$$\Delta_i' \vdash x\, N^1 \cdots N^k \Longrightarrow \mathbb{R}_i : \langle A_i{}^*\rangle$$

for $i = 1 \ldots n$. For claim 3, we observe

$$\begin{array}{ll} \biguplus_{i=1}^n \mathbb{R}_i & \equiv \\ \biguplus_{i=1}^n (x\langle\mathfrak{s}_i'\rangle\, \mathbb{Q}^{i1} \cdots \mathbb{Q}^{ik})\langle A_i{}^*\rangle & \equiv \\ (x\langle\biguplus_{i=1}^n \mathfrak{s}_i'\rangle\, \biguplus_{i=1}^n \mathbb{Q}^{i1} \cdots \biguplus_{i=1}^n \mathbb{Q}^{ik})\langle\biguplus_{i=1}^n \langle A_i{}^*\rangle\rangle & \end{array}$$

where $\mathbb{Q}^{ij} \equiv \biguplus_{q=1}^{m_{ij}} \mathbb{P}_q^{ij}$. So we have

$$\|\biguplus_{i=1}^{n} \mathbb{R}_i\|_L = \max_{j=1}^{k}\{\|\biguplus_{i=1}^{n} \mathbb{Q}^{ij}\|_L\} \le d$$

using (10), thereby proving claim 3.

In case the procedure accepts in step 1(i), we have by induction hypothesis a solution $\mathcal{Y} \equiv N$ with derivable elaborations

$$\Delta_i, x : \mathfrak{s}_i \vdash N \Longrightarrow \mathbb{P}_i : \langle\varphi_i\rangle$$

for $i = 1 \ldots n$, with $(\Delta_i, x : \mathfrak{s}_i)^\circ \subseteq \Gamma_i, x : \sigma_i$ and $(\mathfrak{s}_i \to \varphi_i)^\circ = \sigma_i \to B_i$. It follows that

$$\Delta_i \vdash \lambda x.N \Longrightarrow (\lambda x.\mathbb{P}_i)\langle\mathfrak{s}_i \to \varphi_i\rangle : \langle\mathfrak{s}_i \to \varphi_i\rangle$$

is derivable for $i = 1 \ldots n$, from which claims 1 and 2 are seen to hold. For claim 3, we have by induction hypothesis $\|\biguplus_{i=1}^n \mathbb{P}_i\|_L \le d$, and therefore

$$\|\biguplus_{i=1}^{n}(\lambda x.\mathbb{P}_i)\langle\mathfrak{s}_i \to \varphi_i\rangle\|_L = \|\biguplus_{i=1}^{n}\mathbb{P}_i\|_L \le d$$

thereby proving claim 3.

In case the procedure accepts in step 1(ii), we have by induction hypothesis a solution $\mathcal{Y} \equiv N$ with derivable elaborations

$$\Delta_i \vdash N \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle$$

for $i = 1\ldots n$, with $\Delta_i^\circ \subseteq \Gamma_i$ and $(\mathfrak{s}_i \to \varphi_i)^\circ = \sigma_i \to B_i$. It follows that

$$\Delta_i \vdash \lambda y.N \Longmapsto (\lambda y.\mathbb{P}_i)\langle\mathfrak{s}_i \to \varphi_i\rangle : \langle\mathfrak{s}_i \to \varphi_i\rangle$$

is derivable for $i = 1\ldots n$, from which claims 1 and 2 are seen to hold. For claim 3, we have by induction hypothesis $\|\biguplus_{i=1}^{n}\mathbb{P}_i\|_L \le d$, and therefore

$$\|\biguplus_{i=1}^{n}(\lambda y.\mathbb{P}_i)\langle\mathfrak{s}_i \to \varphi_i\rangle\|_L = \|\biguplus_{i=1}^{n}\mathbb{P}_i\|_L \le d$$

thereby proving claim 3. □

The following is the main lemma for completeness.

Lemma 30. *Suppose $\Delta_i \vdash N \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle$ for $i = 1\ldots n$ with $\|\biguplus_{i=1}^{n}\mathbb{P}_i\| \le d$, where N is a normal form. Then $\mathscr{I}_{\langle d\rangle}(C)$ accepts with*

$$C = \langle\Delta_1{}^\circ \vdash ? : \varphi_1{}^\circ, \ldots, \Delta_n{}^\circ \vdash ? : \varphi_n{}^\circ\rangle$$

Proof: By induction on N.

Notice that $\|\biguplus_{i=1}^{n}\mathbb{P}_i\| \le d$ implies $n \le d$.

In case $N \equiv x$ we have $\Delta_i \vdash x \Longmapsto x\langle\varphi_i\rangle : \langle\varphi_i\rangle$ with $(x : \mathfrak{s}_i) \in \Delta_i$ such that $\mathfrak{s}_i(\varphi_i) > 0$. It evidently follows that $\mathscr{I}_{\langle\bullet\rangle}$ accepts in step 2 (case $k = 0$) from C, and since $n \le d$, so does $\mathscr{I}_{\langle d\rangle}$.

In case $N \equiv \lambda x.N'$ we must have

$$\Delta_i, x : \mathfrak{s}_i \vdash N' \Longmapsto \mathbb{P}'_i : \langle\psi_i\rangle$$

where $\varphi_i = \mathfrak{s}_i \to \psi_i$ for $i = 1\ldots n$. Because $\|\biguplus_{i=1}^{n}\mathbb{P}_i\|_L = \|\biguplus_{i=1}^{n}\mathbb{P}'_i\| \le d$, induction hypothesis applies and shows that $\mathscr{I}_{\langle d\rangle}$ accepts from configuration C' where

$$C' = \langle(\Delta_1, x : \mathfrak{s}_1)^\circ \vdash ? : \psi_1{}^\circ, \ldots, (\Delta_n, x : \mathfrak{s}_n)^\circ \vdash ? : \psi_n{}^\circ\rangle$$

There are two cases. In case *(i)* where there is no other variable y such that $(y : \mathfrak{s}_i{}^\circ) \in \Delta_i{}^\circ$ for all $i = 1\ldots n$, the claim follows easily from induction hypothesis. In case *(ii)* where there is a variable y such that $(y : \mathfrak{s}_i{}^\circ) \in \Delta_i{}^\circ$ for all $i = 1\ldots n$, we note that acceptance from C' implies acceptance from C'' where

$$C'' = \langle\Delta_1{}^\circ \vdash ? : \psi_1{}^\circ, \ldots, \Delta_n{}^\circ \vdash ? : \psi_n{}^\circ\rangle$$

because we can always use $y : \mathfrak{s}_i{}^\circ$ in lieu of $x : \mathfrak{s}_i{}^\circ$, thereby showing the claim.

In case $N \equiv x\, N_1 \cdots N_k$ we must have

$$\Delta_i \vdash x \Longmapsto x\langle\mathfrak{s}_i^1 \to \cdots \to \mathfrak{s}_i^k \to \varphi_i\rangle : \langle\mathfrak{s}_i^1 \to \cdots \to \mathfrak{s}_i^k \to \varphi_i\rangle$$

for $i = 1\ldots n$ with

$$\Delta_i \vdash N_j \Longmapsto \mathbb{Q}_i^j : \mathfrak{s}_i^j$$

for $j = 1\ldots k$ and such that $\mathbb{P}_i \equiv x\langle\ldots\rangle\,\mathbb{Q}_i^1 \ldots \mathbb{Q}_i^k$ (leaving out some annotations for readability). Because we have $\|\biguplus_{i=1}^{n}\mathbb{P}_i\|_L \le d$, it follows that we also have $\|\biguplus_{i=1}^{n}\mathbb{Q}_i^j\| \le d$ for $j = 1\ldots k$. Writing $\mathfrak{s}_i^j = \langle\varphi_{ij}^1, \ldots, \varphi_{ij}^{m_{ij}}\rangle$, it follows from Lemma 15 together with $\Delta_i \vdash N_j \Longmapsto \mathbb{Q}_i^j : \mathfrak{s}_i^j$ for $i = 1\ldots n$ and $\|\biguplus_{i=1}^{n}\mathbb{Q}_i^j\| \le d$ that we have

$$\sum_{i=1}^{n}|\mathfrak{s}_i^j| = \sum_{i=1}^{n} m_{ij} \le \|\biguplus_{i=1}^{n}\mathbb{Q}_i^j\| \le d \text{ for all } j = 1\ldots k$$

By $\sum_{i=1}^{n} m_{ij} \le d$ for all $j = 1\ldots k$ induction hypothesis applies and shows that $\mathscr{I}_{\langle d\rangle}$ accepts from each of the configurations C^j, for $j = 1\ldots k$, where C^j is the configuration

$$\langle\Delta_1{}^\circ \vdash ? : (\varphi_{1j}^1)^\circ, \ldots, \Delta_1{}^\circ \vdash ? : (\varphi_{1j}^{m_{1j}})^\circ, \ldots, \Delta_n{}^\circ \vdash ? : (\varphi_{nj}^1)^\circ, \ldots, \Delta_n{}^\circ \vdash ? : (\varphi_{nj}^{m_{nj}})^\circ\rangle$$

The claim now follows by universal transition in step 2 of $\mathscr{I}_{\langle d\rangle}$. □

Theorem 31 (Soundness and completeness of $\mathscr{A}_{\langle d\rangle}$). *Procedure $\mathscr{A}_{\langle d\rangle}$ is sound and complete for inhabitation in bounded multiset dimension:*

1. *Soundness. If $\mathscr{A}_{\langle d\rangle}(C)$ succeeds with $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$, then $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M.*
2. *Completeness. Whenever $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M, then $\mathscr{A}_{\langle d\rangle}(C)$ succeeds with $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$.*

Proof: We first show that procedure $\mathscr{I}_{\langle d\rangle}$ is sound and complete for inhabitation in bounded multiset dimension, in the following sense:

1. Soundness. If $\mathscr{I}_{\langle d\rangle}(C)$ succeeds with $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$, then $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M.
2. Completeness. Whenever $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M, then $\mathscr{I}_{\langle d\rangle}(C)$ succeeds with $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$.

Soundness is clear from Lemma 29 together with Lemma 22. To prove completeness, suppose $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M. By normalization together with subject reduction in bounded dimension (Theorem 18), there is a normal form N such that $\Delta \vdash N \Longmapsto \mathbb{P} : \mathfrak{s}$ with $\Delta^\circ = \Gamma$, $\mathfrak{s}^\circ = \bigcap_{i=1}^{n} A_i$ and $\|\mathbb{P}\| \le d$. It follows that for some $\mathbb{P}_i$ and φ_i we have $\Delta \vdash N \Longmapsto \mathbb{P}_i : \langle\varphi_i\rangle$ with $\varphi_i{}^\circ = A_i$ and $\|\biguplus_{i=1}^{n}\mathbb{P}_i\| \le d$. Lemma 30 then shows that $\mathscr{I}_{\langle d\rangle}$ accepts from configuration $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$.

We can now prove soundness and completeness of $\mathscr{A}_{\langle d\rangle}$.

Soundness. Assume $\mathscr{A}_{\langle d\rangle}(C)$ succeeds with $C = \langle\Gamma \vdash ? : A_1, \ldots, \Gamma \vdash ? : A_n\rangle$. Then obviously $\mathscr{I}_{\langle d\rangle}(C)$ succeeds. By soundness of procedure $\mathscr{I}_{\langle d\rangle}$, we have $\Gamma \Vdash_d M : \bigcap_{i=1}^{n} A_i$ for some M.

Completeness. Follows immediately from completeness of procedure $\mathscr{I}_{\langle d\rangle}$, since any accepting tree of procedure $\mathscr{I}_{\langle d\rangle}$ with possible repetitions of configurations can obviously be shortened to an accepting tree (without repetitions) of $\mathscr{A}_{\langle d\rangle}$. □

6.3 Complexity of Bounded-Dimensional Inhabitation

By soundness and completeness (Theorem 31) together with subsumption of rank 2-bounded inhabitation (Proposition 23), we are now in a position to prove exponential space completeness of inhabitation in bounded multiset dimension, using the analysis of rank 2-bounded inhabitation by Urzyczyn.

For each $d > 0$ let $INH_{Dim}\langle d\rangle$ denote the decision problem:

- Given Γ and σ, does there exist M such that $\Gamma \Vdash_d M : \sigma$?

The following upper bound argument is analogous to the exponential space upper bound argument for rank 2-bounded inhabitation given in [33].

Proposition 32. *For each $d > 0$, procedure $\mathscr{A}_{\langle d\rangle}$ decides the problem $INH_{Dim}\langle d\rangle$ in* ATIME(N^{2d}) *where N denotes the size of the input Γ and σ.*

Proof: By Theorem 31, procedure $\mathscr{A}_{\langle d\rangle}$ is a semi-decision procedure for $INH_{Dim}(d)$. We now show that procedure $\mathscr{A}_{\langle d\rangle}$ is indeed a decision procedure operating within ATIME(N^{2d}).

Consider the possible distinct configurations of maximal degree d, of the form $\langle\Gamma_1 \vdash ? : A_1, \ldots, \Gamma_d \vdash ? : A_d\rangle$, of procedure $\mathscr{A}_{\langle d\rangle}$. Because there are at most N distinct subterms of types in the input Γ and σ, and no two variables are assigned the same type in all the Γ_i, there are at most N^d distinct assignments of types $\langle\sigma_1, \ldots, \sigma_d\rangle$ to a variable x in $\langle\Gamma_1, \ldots, \Gamma_d\rangle$. The right-hand sides A_i are all subterms of types in the input Γ and σ, hence each right-hand side type can be chosen in at most N ways. Therefore, there are at most N^d possible distinct right-hand side vectors of length d. Consider the (by definition, non-repeating) sequences of configurations along any single path in the computation tree of a run of procedure $\mathscr{A}_{\langle d\rangle}$. After at most N^d consecutive steps, a new variable must be added to the Γ_i to avoid repetition. Since there are at most N^d distinct possible assignments of types to a variable, adding a new variable to the Γ_i can be done at most N^d times avoiding repetition. It follows that the length of paths of non-repeating configurations in the computation tree is bounded by $N^d \times N^d = N^{2d}$. □

Notice that, by Proposition 32, we have decidability in PSPACE (alternating polynomial time) of inhabitation bounded in any fixed dimension d. Generally, bounding dimension by any computable function $f(N)$ yields an upper space bound for inhabitation exponential in $f(N)$.

We turn to the lower bound, which will be by reduction from rank 2-inhabitation [33], via Proposition 23. But first we must establish the following proposition, which is *prima facie* non-obvious due to worst-case exponential blow-up in type size when presenting general intersection types as equivalent strict types in λ^S.

Proposition 33. *Rank* 2 *inhabitation in* λ^S *is* EXPSPACE-*hard.*

Proof: In [33] Urzyczyn proved EXPSPACE-hardness for the halting problem for so-called bus machines by reduction from alternating exponential time Turing-machines. Then, the halting problem for bus machines is reduced to rank 2 inhabitation in [33, Lemma 5]. In particular, it is shown that the following problem is EXPSPACE-hard. Given type constants a_i and type environments $\Gamma_1, \ldots, \Gamma_n$ such that $D := \mathrm{dom}(\Gamma_1) = \ldots = \mathrm{dom}(\Gamma_n)$ and such that for each $x \in D$, $\Gamma_i(x)$ is either a simple type or an intersection of simple types (i.e. normalized intersection type of rank 1), is there a term M such that $\Gamma_i \vdash M : a_i$ for $i = 1 \ldots n$? Note that all occurring types are strict. The claim follows, since the above problem has a solution iff there exists a term M' such that

$$\vdash M' : \bigcap_{i=1}^{n} (\Gamma_i(x_1) \to \ldots \to \Gamma_i(x_m) \to a_i)$$

□

Theorem 34. *The problem* INH_{Dim} *is* EXPSPACE-*complete.*

Proof: Upper bound. By Proposition 32.

Lower bound. By reduction of rank 2-inhabitation in λ^S to INH_{Dim}. Consider a rank 2 instance $\Gamma \vdash_S ? : \sigma$ of the inhabitation problem for λ^S, and let n denote the size of the input Γ and σ. By Proposition 23, we have

$$\exists M.\, \Gamma \vdash_S M : \sigma \Leftrightarrow \exists N.\, \Gamma \Vdash_n N : \sigma$$

Proposition 33 thereby proves the claim. □

We end the paper by recording the following proposition on the complexity of inhabitation in the non-idempotent system of [5]. The proof can be found in [16, Appendix D]. The inhabitation problem in [5] was shown there to be decidable, but complexity was not addressed. The determination of the complexity puts a complexity-theoretic marker (NP vs. EXPSPACE) on the difference between linear, non-idempotent intersection types and the non-linear notion of multiset dimension, as discussed in Section 2 and Section 5. The proof of the NP upper bound is based on the observation that type size imposes a linear bound on the size of minimal inhabitants.

Proposition 35. *The inhabitation problem for non-idempotent intersection types in [5] is* NP-*complete.*

7. Conclusion

We have presented a new notion of dimensionality for the intersection type system based on the idea of elaborations equipped with a norm, and we have applied this concept to obtain a uniform principle for bounding the inhabitation problem with intersection types. We have shown that a multiset interpretation of dimensionality corresponds to the width of simultaneous multiset systems of constraints employed by a sound and complete search procedure for inhabitants. Our main technical result is EXPSPACE-completeness of inhabitation in bounded multiset dimension, and we have shown that this result strictly subsumes the rank 2-bounded fragment, leading to a substantial generalization which is independent of rank or functional order. We believe that the notions of dimension and norm introduced here capture an intuitive idea of "logical width" of intersection types regarded as logical feature vectors.

8. Future Work

We foresee at least three lines of further work which should be immediately enabled by the results presented here. One is the application of our results to synthesis based on inhabitation. We believe that dimensional bound is natural for many applications in this area. If we consider an intersection type as a logical feature vector, it is reasonable to assume that a priori bounds on set theoretic dimensionality can be given relative to the semantics of the application area. This is, for example, rather obvious for applications where intersection types are used to directly express semantic properties, such as are considered in [15, 17] or can be found in abstract interpretation or type refinement. As we have seen, bounding set theoretic dimensionality is not sufficient for decidability, but our notion of norm could allow us to understand how multiset dimensional bound approximates "semantic truth" with respect to inhabitation.

The second area (also discussed in Section 2 and Section 5) is concerned with the connection between our notion of dimensionality and operational (reduction) semantics, and with the relation to systems using concepts of linear logic. In the light of Section 5, it seems reasonable to conjecture that the notions of norm and dimensionality considered here should be systematically related to both topics, and although some observations have been made in the paper, much remains to be clarified. We have also given some results and observations concerning dimensional analysis of other systems, but the deeper meaning of dimension and norm as measures of logical strength should be further investigated. Finally (as also suggested by one of our reviewers), it would be interesting to investigate model theory for the multiset system.

The third area concerns more specific technical questions. Immediate open problems are: Is typability decidable in bounded multiset dimension? Is the problem decidable in bounded set theoretic dimension? It should be noted here by comparison that the system of [5] has a decidable inhabitation problem but an undecidable typability problem. We also briefly touched on the impact of subtyping (Section 5) which is not considered in detail here.

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