BOUNDS ON SCHEDULING WITH LIMITED RESOURCES

by

M. R. Garey and R. L. Graham
Bell Laboratories
Murray Hill, New Jersey 07974

INTRODUCTION

A number of authors (cf. [12],[6], [7],[3],[11],[4],[5],[9]) have recently been concerned with scheduling problems associated with a certain model of an abstract multiprocessing system (to be described in the next section) and, in particular, with bounds on the worst-case behavior of this system as a function of the way in which the inputs are allowed to vary. In this paper, we introduce an additional element of realism into the model by postulating the existence of a set of "resources" with the property that at no time may the system use more than some predetermined amount of each resource. With this extra constraint taken into consideration, we derive a number of bounds on the behavior of this augmented system. It will be seen that this investigation leads to several interesting results in graph theory and analysis.

THE STANDARD MODEL

We consider a system composed of (usually n) abstract identical processors. The function of the system is to execute some given set $\mathcal{J} = \{T_1,\ldots T_r\}$ of tasks. However, $\mathcal{J}$ is partially ordered by some relation* $\prec$ which must be respected in the execution of $\mathcal{J}$ as follows: If $T_i \prec T_j$ then the execution of T_i must be completed before the execution of T_j can begin. To each task T_i is associated a positive real number τ_i which represents the amount of time T_i requires for its execution.

* thus, $<$ is transitive, antisymmetric and irreflexive.

The operation of the system is assumed to be <u>nonpreemptive</u>, which means that once a processor begins to execute a task T_i, it must continue to execute it to completion, τ_i time units later. Finally, the order in which the tasks are chosen is determined as follows: A permutation (or <u>list</u>) $L = \{T_{i_1},\ldots,T_{i_r}\}$ of $\mathcal{J}$ is given initially. At any time a processor is idle, it instantaneously scans L from the beginning and selects the first task T_k (if any) which may validly be executed (i.e., all $T_i \prec T_k$ have been completed) and which is not currently being executed by another processor. Ties by two or more processors for the same task may be broken arbitrarily since the processors are assumed to be identical.

The system begins at time $t = 0$ and starts executing $\mathcal{J}$. <u>The finishing time</u> ω is defined to be <u>the least time at which</u> all tasks have been completed. Of course, ω is a function of L, $\prec$, n and the τ_i. It is known [7] that if $\mathcal{J}' = \{T_1',\ldots,T_r'\}$ with $T_i' \prec T_j' \Longrightarrow T_i \prec T_j$ and $\tau_i' < \tau_i$ for all i and j, and $\mathcal{J}'$ is executed by the system using a list L', then the corresponding finishing time ω' satisfies

$$(0)\qquad \frac{\omega'}{\omega} \le 2 - \frac{1}{n}$$

Furthermore, this bound is best possible. Efficient procedures are known [3],[4], [9] for generating optimal lists when all the τ_i are 1 and <u>either</u> $\prec$ (viewed as a directed graph in the obviously way) is a tree or $n = 2$. However, Ullman [12] has recently shown that even the case of $n = 2$ and $\tau_i \in \{1,2\}$ for all i is polynomial complete* and therefore, probably has no efficient solution in general.

* cf.[10] for a definition of this term.

THE AUGMENTED MODEL

Before proceeding to a description of the new model we first introduce some notation which will make the ensuing discussion mathematically more convenient.

For a given list L, let $F:\mathcal{J} \to 2^{[0,\omega)}$ be defined by $F(T_i) = [\sigma_i, \sigma_i + \tau_i)$ where σ_i is the time at which the execution of T_i was started. Let $f:[0,\omega) \to 2^{\mathcal{J}}$ be defined by $f(t) = \{T_i \in \mathcal{J} : t \in F(T_i)\}$. Thus, $f(t)$ is just the set of tasks which are being executed at time t. The restriction that we have at most n processors can be expressed by requiring $|f(t)| \leq n$ for all $t \in [0,\omega)$.

Assume now that we are also given a set of resources $\mathcal{R} = \{\mathcal{R}_1, \ldots \mathcal{R}_s\}$ and that these resources have the following properties. The total amount of resource $\mathcal{R}_i$ available at any time is (normalized without loss of generality to) 1. For each j, the task T_j requires the use of ρ_{ij} units of resource $\mathcal{R}_i$ at all times during its execution, where $0 \leq \rho_{ij} \leq 1$. For each $t \in [0,\omega)$, let $r_i(t)$ denote the total amount of resource $\mathcal{R}_i$ which is being used at time t. Thus,

$$r_i(t) = \sum_{T_j \in f(t)} \rho_{ij} .$$

In this new model, the fundamental constraint is simply this:

$$r_i(t) \leq 1 \text{ for all } t \in [0,\omega).$$

In other words, at no time can we use more of any resource than is currently available.

The basic problem we shall consider is to what extent the use of different lists for this model can affect the finishing time ω.

SUMMARY OF RESULTS

There are essentially three results which will be proved in this paper. They all are derived from the following situation. We assume we are given a set of tasks $\mathcal{J} = \{T_1, \ldots, T_r\}$, execution times τ_i, a partial order $\prec$ on $\mathcal{J}$, a set of resources $\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_s\}$, task resource usage coefficients* ρ_{ij}, and a positive integer n. Suppose, for two arbitrary lists L and L′, the (augmented) system of n processors executes $\mathcal{J}$ with the resulting finishing times ω and ω', respectively. Note that the use of $n \geq r$ processors is equivalent to having an unlimited number of processors available since clearly there can never be more than r processors active at any given time.

Theorem 1. For $\mathcal{R} = \{\mathcal{R}_1\}$,

$$(1) \qquad \frac{\omega}{\omega'} \leq n .$$

Theorem 2. For $\mathcal{R} = \{\mathcal{R}_1\}$ and $\prec$ empty,

$$(2) \qquad \frac{\omega}{\omega'} \leq 3 - \frac{1}{n} .$$

Theorem 3. For $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\}$, empty and $n \geq r$,

$$(3) \qquad \frac{\omega}{\omega'} \leq s + 1.$$

By way of comparison, the following result is proved in [7].

Theorem 0. For $\mathcal{R} = \emptyset$,

$$\frac{\omega}{\omega'} \leq 2 - \frac{1}{n},$$

Furthermore, as in the case of Theorem 0, examples will be given to show that each of these results is essentially best possible.

Thus, the addition of limited resources into the standard model causes an increase in the worst-case behavior bounds, as might be expected. What is somewhat surprising, however, is the significant effect the partial order $\prec$ can have on these bounds. This is in contrast to the previous case of $\mathcal{R} = \emptyset$ in which the upper bound $\frac{\omega}{\omega'} \leq 2 - \frac{1}{n}$ which holds for arbitrary $\prec$, could, in fact, be achieved by examples with $\prec$ empty. Also significant is the apparent need for somewhat more sophisticated mathematical techniques than were required previously.

PROOF OF THEOREM 1

The proof of (1) is immediate. We merely need to observe that

$$\omega \leq \sum_{i=1}^{r} \tau_i \leq n\omega'$$

since at no time before time ω are all processors idle when using list L, and the number of processors busy at any time never exceeds n.

More interesting is the following example, which shows that (1) is best possible.

* as described in the preceding section.

Example 1

$$T = \{T_1, \ldots, T_n, T_1', \ldots, T_n', \quad \mathcal{R} = \{\mathcal{R}_1$$

$$\tau_i = 1, \ \tau_i' = \varepsilon > 0,$$

$$\mathcal{R}_1(T_i) = \frac{1}{n}, \ \mathcal{R}_1(T_i') = 1, \ 1 \le i \le n.$$

$\prec$ is defined by

$$T_i' \prec T_j \text{ for } 1 \le i \le j \le n.$$

$$L = (T_1, \ldots, T_n, T_1', \ldots, T_n'),$$

$$L' = (T_1', \ldots, T_n', T_1, \ldots, T_n).$$

A simple calculation* shows that

$$\omega = n + n\varepsilon, \quad \omega' = 1 + n\varepsilon.$$

Thus

$$\frac{\omega}{\omega'} = \frac{n+n\varepsilon}{1+n\varepsilon} \to n \text{ as } \varepsilon \to 0$$

which shows that (1) cannot be improved.

PROOF OF THEOREM 2

Let λ denote ordinary Lebesque measure on the real line.** Without loss of generality, we can assume $\omega' = 1$. Hence,

$$\int_0^1 r_1(t)dt \le \omega' = 1$$

where we recall that $r_1(t)$ denotes the total amount of resource $\mathcal{R}_1$ being used at time t.

Suppose $\omega > 3 - \frac{1}{n}$. We shall eventually derive a contradiction. Let $I \le [0,\omega)$ denote the set of times during which all n processors are busy, i.e.,

$$I = \{t\in[0,\omega) : |f(t)| = n\}.$$

Let $\bar{I} = [0,\omega) - I$.

Fact 1. $\lambda(I) \le 1 - \frac{1}{n}$.

To see this, suppose $\lambda(I) > 1 - \frac{1}{n}$. Since $|f(t)| \ge 1$ for all $t\in[0,\omega)$ then

$$\sum_{i=1}^{r} \tau_i \ge n\lambda(I) + 1\cdot(\omega - \lambda(I))$$

$$> (n-1)\left(1 - \frac{1}{n}\right) + \left(3 - \frac{1}{n}\right) = n+1 > n$$

which is impossible since $\omega' = 1$ implies

$$\sum_{i=1}^{n} \tau_i \le n .$$

Thus,

$$\lambda(\bar{I}) = \omega - \lambda(I) > 2 .$$

Fact 2. If $t_1, t_2 \in \bar{I}$ and $t_2 - t_1 \ge 1$ then

$$r_1(t_1) + r_1(t_2) > 1.$$

Proof: Certainly we have $f(t_1) \ne \emptyset$, $f(t_2) \ne \emptyset$ and $f(t_1) \cap f(t_2) = \emptyset$, since $\tau_i \le \omega' = 1$ for all i. Thus, for $T_i \in f(t_2)$, the only reason why it was not executed at time t_1 or sooner must have been because the demand on the resource $\mathcal{R}_1$ would have exceeded the amount available then, i.e.,

$$\mathcal{R}_1(T_i) + r_1(t_1) > 1.$$

But, since $r_1(t_2) \ge \mathcal{R}_1(T_i)$ then (4) holds as asserted

For $t\in[0,\infty)$ define* g(t) by

$$g(t) \equiv \inf\{x : \lambda(\bar{I} \cap [t,x)) = 1\}.$$

Thus, if $g(t) < \infty$ then $\lambda(\bar{I} \cap [t,g(t))) = 1$ and so $g(t) - t \ge 1$. Note that

$$g : [0,g(0)) \to [g(0),\omega)$$

since $\lambda(\bar{I}) > 2$. Therefore

$$1 \ge \int_0^\omega r_1(t)dt \ge \int_{\bar{I}} r_1(t)dt$$

$$\ge \int_{\bar{I}\cap[0,g(g(0)))} r_1(t)dt$$

since $g(g(0)) < \sup \bar{I} =$

$$= \int_{\bar{I}\cap[0,g(0))} r_1(t)dt +$$

$$+ \int_{\bar{I}\cap[g(0),g(g(0)))} r_1(t)dt$$

*The reader will probably find it helpful to construct a timing diagram to understand the behavior of this (and succeeding) examples.

**Since, in all of our applications, the subsets X of $[0,\omega)$ under consideration are finite unions of disjoint half-open intervals, then $\lambda(X)$ is just the sum of the lengths of these intervals.

*where inf $\emptyset$ is defined to be ∞ .

$$= \int_{\bar{I}\cap[0,g(0)]} \Big(r_1(t) + r_1(g(t))\Big)dt$$

since $\frac{dg}{dt} = 1$ a.e.

$$> \int_{\bar{I}\cap[0,g(0))} 1\cdot dt \geq \lambda(\bar{I}\cap[0,g(0))) = 1$$

since we are integrating a step function > 1 over a set of positive measure. Since this is impossible then the assumption that $\omega > 3 - \frac{1}{n}$ is untenable and the theorem is proved.

With a slightly more careful analysis it can be shown that

$$\frac{\omega}{\omega'} < 3 - \frac{3}{n}$$

for n sufficiently large. In the other direction, the following example shows that (2) is not far from best possible.

Example 2.

$$T = \{T_1, T_2, T_1', \ldots, T_{n-1}', T_1'', \ldots, T_{n(n-2)}''\},$$

$$\prec \text{ is empty, } n \geq 2,$$

$$\tau_1 = 1,\ \tau_2 = 1 - \frac{1}{n},\ \tau_i' = \frac{1}{n},\ 1 \leq i \leq n - 1,$$

$$\tau_i'' = \frac{1}{n} - \frac{1}{n^2},\quad 1 \leq i \leq n(n-2),$$

$$\mathfrak{R}(T_1) = \frac{1}{n},\ \mathfrak{R}(T_2) = 1 - \frac{1}{n},\ \mathfrak{R}(T_i') = \frac{1}{n},$$
$$1 \leq i \leq n - 1,$$

$$\mathfrak{R}(T_i'') = 0,\ 1 \leq i \leq n(n-2),$$

$$L = \left(T_1'', \ldots, T_{n(n-2)}'', T_2, T_1', \ldots, T_{n-1}', T_1\right)$$

$$L' = \left(T_1, T_1', \ldots, T_{n-1}', T_2, T_1'', \ldots, T_{n(n-2)}''\right)$$

A straightforward calculation shows that

$$\omega = 3 - \frac{4}{n} + \frac{2}{n^2},\quad \omega' = 1.$$

PROOF OF THEOREM 3

In this case, we assume $\mathfrak{R} = \{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_s\}$, $\prec$ is empty and $n \geq r$. The proof will require several preliminary results. The meaning of undefined terminology in graph theory may be found in [8].

Let G denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. By a valid labeling L of G, we mean a function $L:V \to [0,\infty)$ which satisfies

(5) For all $e = \{a,b\} \in E, L(a) + L(b) \geq 1$.

Define the score of G, denoted by $S(G)$, by

$$S(G) = \inf_{L} \sum_{v\in V} L(v)$$

where the inf is taken over all valid labelings L of G.

Lemma 1. For any graph G, there exists a valid labeling $L:V \to \{0,\frac{1}{2},1\}$ such that

$$S(G)\ \sum_{v\in V} L(v).$$

Proof: For the case of a bipartite graph König's theorem[8] states that the number of edges in a maximum matching equals the point covering number.* Thus, for any bipartite graph G, there exists a valid labeling $L:V \to \{0,1\}$ such that

$$S(G) = \sum_{v\in V} L(v).$$

For an arbitrary graph G, we construct a bipartite graph G_B as follows: For each vertex $v \in V(G)$ we have two vertices $v_1, v_2 \in V(G_B)$; for each edge $\{u,v\} \in E(G)$ we have two edges $\{u_1,v_2\}$ $\{u_2,v_1\} \in E(G_B)$. It is not difficult to verify that $S(G_B) = 2S(G)$ and furthermore, if $L_B:V(G_B) \to \{0,1\}$ is a valid labeling of G_B then $L:V(G) \to \{0,\frac{1}{2},1\}$ by $L(v) = \frac{1}{2}(L(v_1) + L(v_2))$ is a valid labeling of G.

For positive integers m and s, let $G(m,s)$ denote the graph with vertex set $\{0,1,\ldots,(s+1)m-1\}$ and edge set consisting of all pairs $\{a,b\}$ for which $|a-b| \geq m$.

Lemma 2. Suppose $G(m,s)$ is partitioned into s subgraphs** H_i, $1 \leq i \leq s$. Then

$$\max_{1\leq i\leq s} \{S(H_i)\} \geq m. \tag{6}$$

Proof: Assume the lemma is false, i.e., there exists a partition of $G(m,s)$ into H_i, $1 \leq i \leq s$, such that $S(H_i) < m$ for $1 \leq i \leq s$. Thus, by Lemma 1, for each i there exists a valid labeling $L_i:V(H_i) \to \{0,\frac{1}{2},1\}$ such that

$$\sum_{v\in V(H_i)} L_i(v) = S(H_i) < m. \tag{7}$$

---

*i.e., the cardinality of the smallest set of vertices of G incident to every edge of G.

**where $V(H_i) = V(G(m,s))$ and $E(H_i) \subsetneq E(G(m,s))$ for all i.

Let $A = \{a_1 < \ldots < a_p\colon \; L_i(a_j) \le \frac{1}{2}$ for all i, $1 \le i \le s\}$ and let S^* denote

$$\sum_{i=1}^{s} S(H_i).$$

There are three cases.

(i) $p \le m$. In this case we have

$$S^* \ge m\,(s+1) - p \ge m\,(s+1) - m = ms$$

which contradicts (7).

(ii) $m < p \le 2m + 1$. For each edge $\{a_j, a_{m+j}\}$, $1 \le j \le p - m$, there must exist an i such that $L_i(a_j) + L_i(a_{m+j}) \ge 1$. Thus,

$$S^* \ge m(s+1) - p + (p-m) = ms,$$

again contradicting (7).

(iii) $p > 2m + 1$. We first note that for each vertex $v \in V(G(m,s))$, there exists an i such that $L_i(v) \ge \frac{1}{2}$. For suppose $L_i(v) = 0$ for $1 \le i \le s$. Then there must be some a_j such that $|a_j - v| \ge m$. But since $L_i(a_j) \le \frac{1}{2}$ for all i, then $L_i(a_j) + L_i(v) \le \frac{1}{2}$ for all i which is a contradiction.

For each i, let n_i denote the number of vertices v such that $L_i(v) = 1$. Then

$$|\{v\colon L_i(v) > 0\}| \le 2m - 1 - n_i$$

since otherwise,

$$\sum_{v \in V(H_i)} L_i(v) \ge n_i \cdot 1 + (2m - 2n_i)\cdot\tfrac{1}{2} = m$$

which contradicts (7). Therefore

$$(8) \quad \sum_{i=1}^{s} |\{v\colon L_i(v) > 0\}| \le (2m-1)s - \sum_{i=1}^{s} n_i$$

Let q denote the number of vertices v such that there is exactly one i for which $L_i(v) > 0$. Then

$$(9) \quad \sum_{i=1}^{s} |\{v\colon L_i(v) > 0\}| \ge 2(m(s+1) - q) + q.$$

Combining (8) and (9)

$$(10) \qquad q \ge 2m + s + \sum_{i=1}^{s} n_i$$

Of course, we may assume without loss of generality that if $L_i(v) = 1$ then $L_j(v) = 0$ for all $j \ne i$. Hence, by the definition of n_i, there must be at least $2m + s$ vertices, say, $b_1 < \ldots < b_{2m+s}$, such that

$$\sum_{i=1}^{s} L_i(b_j) = \tfrac{1}{2}, \text{ i.e.}$$

for each b_j there is a unique L_i such that $L_i(b_j) = \frac{1}{2}$ and $L_k(b_j) = 0$ for all $k \ne i$. Thus, if $|b_j - b_k| \ge m$ then for some i $L_i(b_j) = L_i(b_k) = \frac{1}{2}$. Since $|b_1 - b_{2m+s}| \ge m$, let i_0 be such that $L_{i_0}(b_1) = L_{i_0}(b_{2m+s}) = \frac{1}{2}$. But, by the same reasoning we must also have $L_{i_0}(b_{m+j}) = L_{i_0}(b_1) = \frac{1}{2}$ and $L_{i_0}(b_{2m+s}) = L_{i_0}(b_j) = \frac{1}{2}$ for $1 \le j \le m + s$. Therefore

$$S(H_{i_0}) = \sum_{v \in V(H_{i_0})} L_{i_0}(v) \ge (2m+s)\cdot\tfrac{1}{2} \ge m$$

which is a contradiction. This completes the proof of Lemma 2.

Recall that when $\mathcal{J}$ is executed using the list L, $F(T_i)$ is defined to be the interval $[\sigma_i, \sigma_i + \tau_i)$ where σ_i is the time at which T_i starts to be executed and $\sigma_i + \tau_i$ is the time at which T_i is finished. Note that because of the way in which the operation of the system is defined, each σ_i is a sum of a subset of the τ_j's.

We may assume without loss of generality that $\omega' = 1$. Assume now that $\omega > s + 1$. Furthermore, suppose each τ_i can be written as

$$\tau_i = \frac{k_i}{m} \text{ where } k_i \text{ is a}$$

positive integer. Thus, $k_i \le m$, since $\tau_i \le \omega' = 1$. Also, for $1 \le i \le s$, each $r_i(t)$ is constant on each interval $\left[\frac{k}{m}, \frac{k+1}{m}\right]$, this value being $r_i\!\left(\frac{k}{m}\right)$. An important fact to note is that since $\prec$ is empty and $n \ge r$ then, for $t_1, t_2 \in [0, \omega)$ with $t_2 - t_1 \ge 1$, we must have

$$\max_{1 \le i \le s} \{r_i(t_1) + r_i(t_2)\} > 1.$$

For otherwise, any task being executed at time t_2 should have been executed at time t_1 or sooner. Thus, for each i, $1 \le i \le s$, we can construct a graph H_i as follows:

$$(11) \quad \begin{aligned} &V(H_i) = \{0, 1, \ldots, (s+1)m - 1\}\ ; \\ &\{a,b\} \text{ is an edge of } H_i \text{ iff} \\ &r_i\!\left(\frac{a}{m}\right) + r_i\!\left(\frac{b}{m}\right) > 1. \end{aligned}$$

Note that if $|a-b| \geq m$ then $\{a,b\}$ is an edge of at least one H_i, $1 \leq i \leq s$.

Hence, it is not difficult to see that $G_m \subseteq \bigcup_{i=1} H_i$. Note that by (11), the mapping $L_i: V(H_i) \to [0,\infty)$ defined by $L_i(a) = r_i(\frac{a}{m})$ is a valid labeling of H_i. Since $G \subseteq G'$ implies $S(G) \leq S(G')$ and the condition on the r_i in (11) is a strict inequality then by Lemma 2 it follows that

$$\max_i \left\{ \sum_{k=0}^{(s+1)m-1} r_i\left(\frac{k}{m}\right) \right\} = \max_i \left\{ \sum_{v \in V(H_i)} L_i(v) \right\} \tag{12}$$
$$> \max_i \{S(H_i)\} \geq m.$$

But as we have already remarked, we must have

$$\frac{1}{m} \sum_{k=0}^{(s+1)m-1} r_i\left(\frac{k}{m}\right) \leq \int_0^{\omega} r_i(t)dt \leq 1, \tag{13}$$
$$1 \leq i \leq s,$$

i.e.,

$$\sum_{k=0}^{(s+1)m-1} r_i\left(\frac{k}{m}\right) \leq m, \ 1 \leq i \leq s.$$

This is a <u>contradiction</u> and Theorem 3 is proved in the case that $\tau_i = \frac{k_i}{m}$ where k_i is a positive integer for $1 \leq i \leq r$. Of course, it follows immediately that Theorem 3 holds when all the τ_i are <u>rational</u>. The proof of Theorem 3 will be <u>completed</u> with the establishment of the following lemma.

<u>Lemma 3.</u> Let $\tau = (\tau_1, \ldots, \tau_r)$ be a sequence of positive real numbers. Then, for any $\varepsilon > 0$, there exists $\tau^* = (\tau_1^*, \ldots, \tau_r^*)$ such that:

(i) $|\tau_i^* - \tau_i| < \varepsilon$ for $1 \leq i \leq r$;

(ii) For all $S, T \subseteq \{1, \ldots, r\}$,

$$\sum_{s \in S} \tau_s \leq \sum_{t \in T} \tau_t \text{ if and only if}$$

$$\sum_{s \in S} \tau_s^* \leq \sum_{t \in T} \tau_t^* ;$$

(iii) All τ_i^* are positive rational numbers.

<u>Remark:</u>

The importance of (ii) is that it guarantees that the <u>order</u> of execution of the T_i using the <u>list</u> L is the same for τ and τ^*. Thus, if L is used to execute $\mathcal{J}$, once using execution times τ_i and once using execution times τ_i^* then the corresponding finishing times ω and $\tilde{\omega}$ satisfy

$$|\omega - \omega^*| \leq r\varepsilon.$$

Hence, if there were an example $\mathcal{J}$ with $\frac{\omega}{\omega'} > s + 1$ and some of the τ_i irrational, then we could construct another example $\mathcal{J}^*$ by slightly changing the τ_i to rational τ_i^* so that the corresponding new finishing times ω^* and ω'^* satisfy

$$|\omega - \omega^*| \leq r\varepsilon, \ |\omega' - \omega'^*| \leq r\varepsilon$$

and, therefore, if ε is sufficiently small, we still have

$$\frac{\omega^*}{\omega'^*} > s + 1.$$

However, this would contradict what has already been proved. Lemma 3 is implied by the following slightly more general result. The proof we give here is due to V. Chvatal (personal communication).

<u>Lemma 3'</u> Let S denote a finite system <u>of inequalities</u> of the form

$$\sum_{i=1}^{r} a_i x_i \geq a_0 \text{ or } \sum_{i=1}^{r} a_i x_i > a_0$$

where the a_i are rational. Then, for any $\varepsilon > 0$, if S has a real solution $(x_1, \ldots x_r)$ then S has a <u>rational</u> solution $(x_1^*, \ldots, x_r^*)$ with $|x_i - x_i^*| < \varepsilon$ for all i.

<u>Proof:</u> We proceed by induction on r. For <u>r = 1</u> the result is immediate. Now, let S be a system of inequalities in $r > 1$ variables which is solvable in reals. S splits into two classes: S_0, the subset of inequalities not involving x_r, and S_1, the subset of inequalities involving x_r. Each inequality in S_1 can be written in one of the following four ways:

(a) $\alpha_0 + \sum_{i=1}^{r-1} \alpha_i x_i \leq x_r$,

(b) $\alpha_0 + \sum_{i=1}^{r-1} \alpha_i x_i < x_r$,

(c) $\beta_0 + \sum_{i=1}^{r-1} \beta_i x_i \geq x_r$,

(d) $\beta_0 + \sum_{i=1}^{r-1} \beta_i x_i > x_r$.

For each pair of inequalities, one of type (a) and one of type (c), we shall consider the inequality

$$\text{(e)} \quad \alpha_0 + \sum_{i=1}^{r-1} \alpha_i x_i \le \beta_0 + \sum_{i=1}^{r-1} \beta_i x_i .$$

Similarly, the pairs of types $\{(a),(d)\}$, $\{(b),(c)\}$ and $\{(b),(d)\}$ give rise to inequalities

$$\text{(f)} \quad \alpha_0 + \sum_{i=1}^{r-1} \alpha_i x_i < \beta_0 + \sum_{i=1}^{r-1} \beta_i x_i .$$

Let S^* be the set of all inequalities of type (e) and (f) we obtain from S_1. Since by hypothesis, $S = S_0 \cup S_1$ has a real solution $(x_1,\ldots,x_r)$ then $S_0 \cup S^*$ has the real solution $(x_1,\ldots,x_{r-1})$. But $S_0 \cup S^*$ only involves $r - 1$ variables so that, by the induction hypothesis, $S_0 \cup S^*$ has a rational solution $(x_1^*,\ldots,x_{r-1}^*)$ with $|x_i - x_i^*| < \varepsilon'$ for all i and any preassigned $\varepsilon' > 0$. Substituting the x_i^* into (a),(b), (c) and (d) we obtain a set of inequalities

$$\text{(g)} \quad a^* \le x_r, \ b^* < x_r, \ c^* \ge x_r, \ d^* > x_r$$

where the a^*, b^*, c^* and d^* are rational. Since the x_i satisfy (e) and (f), we have $a^* \le c^*$, $b^* < c^*$, $a^* < d^*$, $b^* < d^*$. Thus, for any $\varepsilon > 0$, if ε' is chosen to be suitably small, then there is a rational x_r^* satisfying (g) and with $|x_r - x_r^*| < \varepsilon$, completing the proof of Lemma 3'. This proves Lemma 3, and hence, Theorem 3.

The following example shows that the bound in Theorem 3 cannot be impoved.

<u>Example 3</u>

$$\mathcal{J} = \{T_1, T_2, \ldots, T_{s+1}, T_1', T_2', \ldots, T_{sN}'\}$$

$$\prec = \emptyset, \quad n \ge s(N+1) + 1 = r,$$

$$\tau_i = 1, \quad 1 \le i \le s + 1, \ \tau_i' = 1/N, \ 1 \le i \le sN,$$

$$R_i(T_i) = 1 - \frac{1}{N}, \ R_i(T_j) = \frac{1}{sN}, \ j \ne i, \ 1 \le i \le s,$$

$$R_i(T_j') = \frac{1}{N}, \ 1 \le j \le sN, \ 1 \le i \le s,$$

$$L = (T_1, T_1', \ldots, T_N', T_2, T_{N+1}', \ldots, T_{k+1}, T_{kN+1}', T_{kN+2}', \ldots, T_{(k+1)N}', T_{k+2}, \ldots, T_{sN}', T_{s+1})$$

$$L' = (T_1', T_2', \ldots, T_{sN}', T_1, T_2, \ldots, T_{s+1}).$$

It is easily checked that for this case

$$\omega = s + 1, \quad \omega' = 1 + \frac{s}{N}$$

so that ω/ω' is arbitrarily close to $s + 1$ for N sufficiently large.

<u>Concluding Remarks.</u>

The results which have been discussed in this paper lead naturally to a number of possible extensions, several of which we mention here.

We first note that for the case $\mathcal{R} = \{\mathcal{R}_1\}$, $n \ge r$, and general $\prec$, example 1 may be used to show that $\frac{\omega}{\omega'}$ can be arbitrarily large.

Regarding Lemma 1, an algorithm can be given which determines $S(G)$ (and a corresponding valid labeling as well) in at most

$$O\left(|E| \sqrt{|V|}\right)$$

operations. A similar algorithm may be used for the following dual problem: Given a graph G, determine

$$\max_{L^*} \sum_{e \in E} L^*(e)$$

where the max ranges over all functions $L^*: E \to [0,\infty)$ such that for all $v \in V$,

$$\sum_{e' \in E(v)} L^*(e') \le 1$$

where $E(v)$ is the set of all edges incident to v. It would be interesting to investigate the analogous questions for hypergraphs.

The following result follows more or less directly from Lemma 2:

<u>COROLLARY</u>: For a positive integer n, let $f_i: [0,n+1) \to [0,\infty)$, $1 \le i \le n$, be (Lebesque) measurable functions satisfying

(i) If $t_1, t_2 \in [0,n+1)$ with $|t_1 - t_2| \ge 1$ then

$$\max_{1 \le i \le n} \{f_i(t_1) + f_i(t_2)\} \ge 1.$$

Then

$$\text{(ii)} \quad \max_{1 \le i \le n} \int_{[0,n+1]} f_i \, d\lambda \ge 1$$

It is interesting to note that, at present, no purely analytical proof of the Corollary is known.

The techniques of Lemma 2 may also be used to derive several new results in graph theory. In particular, it follows that if m is a positive integer and G_m denotes the graph with vertex set $V_m = \{0,1,\ldots,3m-1\}$ and edge set $E_m = \{\{a,b\} \subseteq V_m : \min\{a-b, 3m-a+b\} \ge m\}$ then <u>any</u> 2-coloring of E_m contains m disjoint edges having the same color.

The corresponding general conjecture is that for a fixed $s \ge 1$, $V_m = \{0,1,\ldots,(s+1)m-1\}$ and $E_m = \{\{a,b\} \subseteq V_m : \min\{a-b, (s+1)m-a+b\} \ge m\}$ and it is required to show that any s-coloring of E_m contains m disjoint edges having the same color. At present, this conjecture is still open. If true, it is close to being best possible since there exist s-colorings of the edges of the complete graph on (s+1) m - s vertices which have no set of m disjoint edges having a single color (cf.[1],[2]).

Finally, it is natural to inquire under what restrictions do there exist efficient algorithms for determining optimal schedules for problems of the type considered herein(e.g.,cf.[6],[12]).

The authors take pleasure in acknowledging the resourceful suggestions of S. A. Burr, J. C. Cheng, V. Chvátal, D. S. Johnson, Shen Lin, and H. S. Witsenhausen. The introduction of limited resources into scheduling problems of this type was first suggested by E. Arthurs.

REFERENCES

[1] Burr, S. A., Generalized Ramsey Theory for Graphs - A Survey of Recent Results (to appear).

[2] Cockayne, E. J. and Lorimer, P. J. The Ramsey Graph Numbers for Stripes Univ. of Auckland Math. Dept. Report series No. 37, April (1973).

[3] Coffman, E. G. and Graham, R. L., Optimal Scheduling for Two-processor Systems, Acta Informatica 1 No. 3 (1972) 200-213.

[4] Fujii, M., Kasami, T. and Ninomiya, K., Optimal Sequencing of Two Equivalent Processors, SIAM J. of Appl. Math. 17 No. 3 (1969) 784-789.

[5] ________, Erratum SIAM J. of Appl. Math. 20 No. 1 (1971) 141.

[6] Graham, R. L., Bounds on Multiprocessing Anomalies and Related Packing Algorithms, AFIPS Conf. Proc. 40 (1972) 205-217.

[7] ________, Bounds on Multiprocessing Timing Anomalies, SIAM J. of Appl. Math. 17 No. 2 (1969) 263-269.

[8] Harary, F., Graph Theory, Addison-Wesley, Reading (1969).

[9] Hu, T. C., Parallel Scheduling and Assembly Line Problems, Oper. Res.9 No. 6 (1961) 841-848.

[10] Karp, R. M. Reducibility Among Combinatorial Problems, in Complexity of Computer Computations, Plenum, New York (1972)

[11] Kaufman, M. T., Anomalies in Scheduling Unit-time Tasks, Stanford Electronics Laboratories Tech. Rep. No. 34 June (1972).

[12] Ullman, J. D., Polynomial Complete Scheduling Problems, Univ. of Calif. (Berkeley) Computer Science Tech. Rep. 9, March (1973).