Integer programming

and

totally unimodular matrices

E.de Klerk
Delft University of Technology

Week 7: Slides about Chapter 8 of the Lecture notes.

The matching polytope: representation

Recall from week 2:

Theorem: Let $G = (V, E)$ be bipartite. Then $P_{matching}(G)$ is the set of vectors $x \in \mathbb{R}^{|E|}$ which satisfy:

$$\begin{aligned} x_e &\geq 0 \text{ for each } e \in E; \\ \textstyle\sum_{e \ni v} x_e &\leq 1 \text{ for each } v \in V \end{aligned}$$

Proof via Exercises 3.11 and 3.26.

Maximal weighted matching as linear optimization problem:

$$\max_{x \in \mathbb{R}^{|E|}} w^T x$$

subject to the above constraints.

Integer program:

$$\max\left\{c^T x \mid Ax = b,\ x \text{ integer}\right\}$$

Example

Finding maximal cardinality matching. Define the *incidence matrix*:

$$A_{v,e} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e. \end{cases}$$

Size of the maximal matching:

$$\nu(G) = \max\left\{\mathbf{1}^T x \mid Ax \leq \mathbf{1},\ x \geq 0,\ x \text{ integer}\right\}$$

Recall from Exercise 3.26, 3.27 that we can drop the integrality condition for *bipartite graphs*.

We can do this because all the vertices of the *matching polytope* are *integer* for *bipartite graphs*.

Totally unimodular matrices

A matrix A is *totally unimodular* if every *square submatrix* has determinant 0, 1, or -1.

NB: Every entry of A is 0, 1, or -1.

Theorem: Let A be totally unimodular and b an integer vector. The polytope

$$P := \{x \mid Ax \leq b\}$$

is integer (all vertices are integer).

Proof:

Let A be $m \times n$. Every vertex z is determined by some $n \times n$ nonsingular submatrix A' in the sense $A'z = b'$ where b' contains the corresponding rows of b. Now use Cramer's rule to solve for z.

Total unimodularity and bipartite graphs

Theorem: A graph is bipartite if and only if its incidence matrix is totally unimodular.

Proof:
'$\Leftarrow$': Let A totally unimodular. Assume G not bipartite then G contains an odd cycle (why?). The submatrix of A which corresponds to the odd cycle has determinant 2; this contradicts the total unimodularity of A.

'$\Rightarrow$': Let G be bipartite. Consider a $t \times t$ submatrix of A. Proof by induction on t. ($t = 1$ follows from the definition of the incidence matrix.)

Corollaries: König's theorems

König Matching Theorem: Let G be bipartite. Cardinality of maximal matching $=$ cardinality of minimal vertex cover ($\nu(G) = \tau(G)$).

Proof: We know that the matching number is given by

$$\nu(G) = \max \left\{ \mathbf{1}^T x \mid Ax \leq \mathbf{1},\ x \geq 0 \right\}.$$

Dual problem:

$$d^* = \min \left\{ \mathbf{1}^T y \mid y \geq 0,\ A^T y \geq \mathbf{1} \right\}.$$

By LP duality $d^* = \nu(G)$.

Let y be the *incidence vector* of the minimal vertex cover. Then y is feasible in the dual problem (why?) with objective value $\tau(G)$. Assume y not optimal. Then there is some $0-1$ optimal solution (why?) with objective value $d^* < \tau(G)$. This $0-1$ solution must be the incidence vector of a vertex cover. Contradiction.

Corollaries: König's theorems (ctd)

König Edge Cover Theorem: Let G be bipartite. Cardinality of maximal coclique $=$ cardinality of minimal edge cover ($\alpha(G) = \rho(G)$).

Proof:

Same as previous proof: only replace A by A^T.

Incidence matrix of a directed graph

Let $D = (V, A)$ a directed graph.

Define the $|V| \times |A|$ *incidence matrix*:

$$M_{v,a} = \begin{cases} 1 & \text{if } a \text{ leaves } v, \\ -1 & \text{if } a \text{ enters } v, \\ 0 & \text{otherwise} \end{cases}$$

NB: Every column of M has exactly one 1 and one -1, the other elements are zeroes.

Theorem: The incidence matrix M of a directed graph D is totally unimodular.

Proof:
Let B be a $t \times t$ square submatrix of M. Proof by induction on t (case $t = 1$ trivial). Three cases:

- **Case 1:** B has a zero column $\Rightarrow \det(B) = 0$.
- **Case 2:** B has a column with exactly one 1. Calculate $\det(B)$ using this column and use the induction assumption.
- **Case 3:** Every column of B has one 1 and one -1. The row vectors of B add up to the zero vector $\Rightarrow \det(B) = 0$.

Flows and total unimodularity

Given: directed graph $D = (V, A)$.

Recall from week 3: A function $x : A \mapsto \mathbb{R}$ is called an $r - s$ *flow* if:

$$x(a) \geq 0 \quad \forall a \in A$$

and

$$\sum_{a \in \delta^{in}(v)} x(a) = \sum_{a \in \delta^{out}(v)} x(a) \quad \forall v \in V \setminus \{r, s\},$$

Second condition is *flow conservation*: in-flow = out-flow in each node.

In terms of the incidence matrix M, the flow conservation can be written as:

$$M'x = 0,$$

where M' is obtained from M by deleting rows r and s.

Flows and total unimodularity (ctd)

We can now prove the ‘max $r - s$ flow = min capacity $r - s$ cut’ theorem. Let w^T be row r of M, and M' the matrix obtained by removing the rows corresponding to r and s.

Maximal $r - s$ flow given by:

$$\max\left\{w^T x \mid x \leq c,\ M'x = 0,\ x \geq 0\right\},$$

where c is the *integer capacity vector* for the arcs.

Dual problem:

$$\min\left\{y^T c \mid M'^T z + y \geq w,\ y \geq 0\right\}.$$

Rewrite dual constraints:

$$\begin{pmatrix} M'^T & I \\ 0 & I \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \geq \begin{pmatrix} w \\ 0 \end{pmatrix}$$

Coefficient matrix is TUM (why?)

Flows and total unimodularity (ctd)

Dual problem has integer solution y, z because the coefficient matrix is TUM. Define

$$W := \{v \in V \setminus \{r, s\} \mid z_v \leq -1\} \cup \{r\}.$$

We now show that

$$c(\delta^{out}(W)) = y^T c \; (= \text{ max flow})$$

by showing

$$\text{if } a = (u, v) \in \delta^{out}(W) \text{ then } y_a \geq 1. \tag{1}$$

Expand the vector z to include r, s by defining:

$$\tilde{z}_v = \begin{cases} -1 & \text{if } v = r \\ 0 & \text{if } v = s \\ z_v & \text{otherwise} \end{cases}$$

Now $M^T \tilde{z} + y \geq 0$. If $a = (u, v) \in \delta^{out}(W)$ then $y_a + \tilde{z}_u - \tilde{z}_v \geq 0$ which implies (1).

Interval matrices

An *interval matrix* has $0-1$ entries and each row is of the form

$$(0, \ldots, 0, \underbrace{1, \ldots\ldots\ldots, 1}_{\text{consecutive 1's}}, 0, \ldots, 0).$$

Theorem: Each interval matrix A is totally unimodular.

Proof: Let B be a $t \times t$ submatrix of A and let

$$N := \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Note that $\det(N) = 1$. Now NB^T is (a submatrix of) the incidence matrix of some directed graph (why?) Therefore NB^T is totally unimodular and

$$\{0, \pm 1\} \ni \det(NB^T) = \det(N)\det(B^T) = \det(B).$$