Available online at www.sciencedirect.com

SCIENCE DIRECT®

Computer Aided Geometric Design 22 (2005) 103–119

COMPUTER AIDED GEOMETRIC DESIGN

www.elsevier.com/locate/cagd

s-power series: an alternative to Poisson expansions for representing analytic functions

J. Sánchez-Reyes *, J.M. Chacón

Department of Applied Mechanics, University of Castilla-La Mancha, ETS Ingenieros Industriales, Campus Universitario, 13071, Ciudad Real, Spain

Received 20 September 2003; received in revised form 20 September 2004; accepted 25 September 2004

Available online 8 October 2004

Abstract

Morin and Goldman [Computer Aided Geometric Design 17 (2000) 813] have recently presented a remarkable new framework, based on employing Poisson series, for describing analytic functions in CAD. We compare this Poisson formulation with s-power series, modified Newton series that can be regarded as the two-point analogue of Taylor expansions. Such s-power series yield, over finite intervals, better approximations for CAD purposes, as they are polynomial and hence expressible in the Bernstein–Bézier standard, can be pieced together in a smooth Hermitian spline and, in general, display better convergence.

Keywords: Hermite approximation; Hermitian spline; Newton series; s-power series; Poisson function; Two-point Taylor expansion

1. Introduction: the representation of analytic functions in CAGD

The parametric representation of curves and surfaces in CAGD employs polynomial or rational functions (Farin, 2001; Piegl and Tiller, 1997). However, analytic functions (Davis, 1975) provide a richer framework, needed for the exact representation of transcendental, i.e., nonalgebraic curves or surfaces (Lawrence, 1972), not encompassed by the standard rational model. In addition, some geometry processing operations, such as offsetting or computing arc lengths, involve non-polynomial functions, in particu-

* Corresponding author.
E-mail addresses: javier.sanchezreyes@uclm.es (J. Sánchez-Reyes), jesusmiguel.chacon@uclm.es (J.M. Chacón).

0167-8396/$ – see front matter
doi:10.1016/j.cagd.2004.09.003

lar square roots; hence the interest of a general tool for representing and manipulating analytic functions in an efficient way.

A first option is provided by the concept of *B-basis* (Peña, 1999), which carries over the customary Bernstein basis to general functional spaces of finite dimension. Given a space with normalized totally positive bases, among all them there exists a unique basis, called B-basis, with optimal shape-preserving properties. Intuitively, if a space has a B-basis we can define curves through a control polygon in a similar way to our familiar Bézier case. These curves display all the positive geometric properties of the Bézier scheme (variation diminishing, containment in the convex-hull, affine invariance, tangency to the control polygon at the endpoints) plus optimal stability and the existence of a de Casteljau-type algorithm. However, each space of functions has a particular B-basis, and the number of basis functions required equals the dimension of the space. In consequence, we cannot represent arbitrary analytic functions in a unified manner.

The obvious alternative is using an infinite number of basis functions, that is, some kind of expansion. Morin and Goldman (2000a, 2000b, 2001a, 2001b) investigated on Poisson series and derived a model that can be regarded as the analogue of the polynomial Bézier scheme for representing analytic functions on semi-infinite intervals. The coefficients of the series are endowed with a geometric meaning as control points, defining a control polygon that enjoys the positive geometric properties of the Bézier formulation. Moreover, the algorithms for Poisson curves admit an interpretation in terms of an analytic blossom. Sánchez-Reyes (1997, 2000) explored the two-point analogue of Taylor expansions, dubbed *s-power series*, which also admit a representation in terms of a control polygon. Truncating at the kth term the s-power series furnishes the order-k Hermite interpolant, i.e., the degree-$(2k+1)$ polynomial curve that reproduces up to the kth derivative of the original function at the endpoints of a given interval. By piecing these approximations we obtain a C^k Hermitian spline (Grisoni et al., 1999).

Other polynomial expansions over finite intervals have been employed in a CAD context. For instance, Legendre series are advocated for approximating offset curves (Li and Hsu, 1998), the inversion of polynomial functions (Farouki, 2000), or degree-reduction (Lee et al., 2002).

In this article we compare Poisson and s-power series for approximating analytical functions over finite intervals. In Sections 2 and 3, we revisit Poisson and s-power series, respectively, with special emphasis on their converge and techniques to avoid singularities. Section 4 explains through several comparative examples why s-power series display better convergence than Poisson series, which leads to the conclusions drawn in Section 5.

2. Poisson functions: the analogue of Bézier functions on semi-infinite intervals

2.1. Poisson series as a limiting case of nonparametric Bézier curves

Given a function $f(t)$, its Poisson expansion around $t=0$ is defined as:

$$f(t)=\sum_{k=0}^{\infty} p_k b_k(t), \quad b_k(t)=\frac{\mathrm{e}^{-t}t^k}{k!}. \tag{1}$$

The Poisson coefficients p_k are easily computed (Morin and Goldman, 2000a) as those of the Taylor expansion for the function $\mathrm{e}^t f(t)$ around $t=0$.

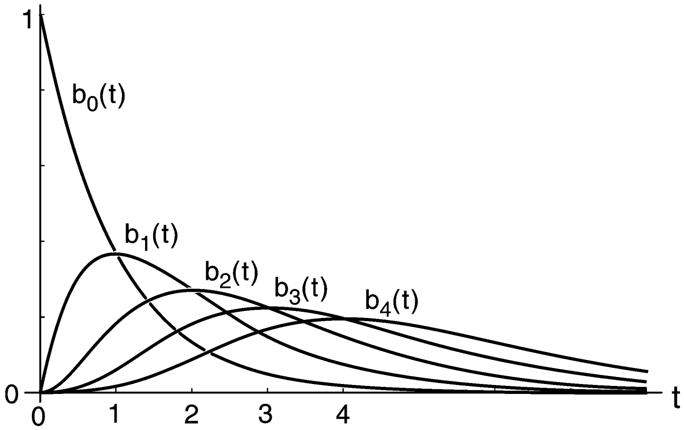

Fig. 1. Poisson basis functions $b_k(t)$, $k = 0, \dots, 4$.

$$p_k = g^{(i)}(0) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(0), \quad g(t) = \mathrm{e}^t f(t). \tag{2}$$

The *Poisson basis functions* $b_k(t)$ (1), plotted in Fig. 1, are nonnegative over the positive semi-axis $t \in [0, \infty)$ and form a partition of unity. Such functions can be regarded as the limiting case $n \to \infty$ of degree-n Bernstein polynomials, defined over $t \in [0, n]$. Hence, the summation (1) admits an elegant geometric interpretation, as the limit case $n \to \infty$ of a degree-n nonparametric Bézier curve over $t \in [0, n]$. This nonparametric curve has control points $\mathbf{p}_k = (k, p_k)$, i.e., integer Bézier abscissas $\{0, 1, 2, \dots\}$ regularly spaced along the x-axis, defining an infinite control polygon that inherits the positive properties of the Bézier scheme:

- It mimics the shape of $f(t)$, now in a right neighbourhood of the origin $t = 0$. Points $\mathbf{p}_k$ have a push/pull effect, since $b_k(t)$ enjoys unimodality, attaining its maximum precisely at $t = k$.
- The polygon is tangent to $f(t)$ at $\mathbf{p}_0$.
- If the series converges on $[0, \infty)$, the convex hull and variation diminishing properties hold.
- Linear precision: a linear function $f(t)$ has control points with ordinates $p_k = f(k)$.

A representation (2) of a function that enjoys convergence in a neighbourhood of the origin is called *Poisson function*. The definitions extend in a straightforward manner to the parametric case, where the Poisson scheme enjoys affine invariance.

2.2. Subdivision

Suppose that we want a new representation of $f(t)$, with control points $\mathbf{q}_k = (k\rho, q_k)$ over abscissas $\{0, \rho, 2\rho, \dots\}$. Clearly, this is tantamount to scaling the graph and polygon of $f(\rho t)$ a factor ρ along the x-axis. In consequence, the new ordinates q_i are those of $f(\rho t)$, and the new basis functions are $b_k(t/\rho)$, so the Poisson series is rewritten as:

$$f(t) = \sum_{k=0}^{\infty} q_k b_k(t/\rho), \quad q_k = \sum_{i=0}^{k} \binom{k}{i} \rho^i f^{(i)}(0). \tag{3}$$

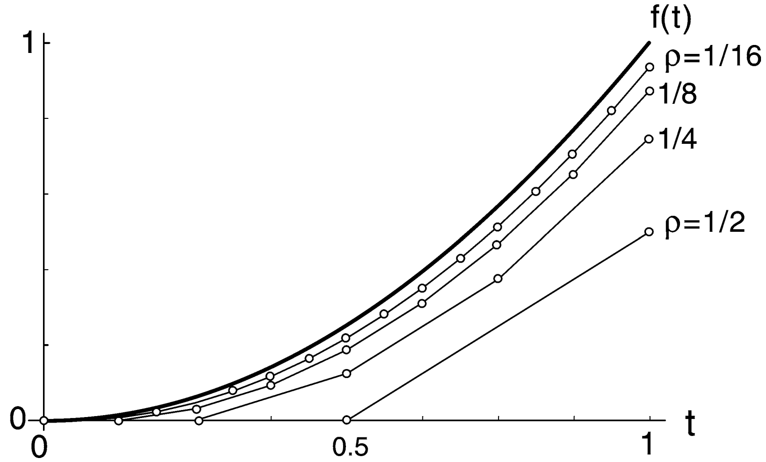

Fig. 2. Successive Poisson control polygons for $f(t) = t^2$, $t \in [0, 1]$.

In the polynomial setting, DeRose (1988) noted the equivalence between subdivision and this linear reparameterization. Morin and Goldman (2001a) also observe that the new kth ordinate q_k (3) derives from the original ones $\{p_i\}_{i=0}^{k}$ (2) via a de Casteljau-type algorithm (left-sided subdivision) using a simple summation. In the case $0 < \rho < 1$ (refined representation), a repeated subdivision furnishes a sequence of control polygons that converges to $f(t)$ over the convergence interval of the series. Therefore, such polygons provide a piecewise linear approximation to $f(t)$. This property is illustrated in Fig. 2 by the function $f(t) = t^2$, where we have chosen successive powers $\rho = 2^{-i}$, $i = 1, 2, 3, 4$. Hermann (2002) showed that this converge of the refined polygon holds for a certain family of bases that includes the Bernstein and Poisson bases.

If we seek a representation of $f(t)$ over an arbitrary interval $[t_0, \infty)$, rather than on $[0, \infty)$, with control points over abscissas $\{t_0, t_0 + 1, t_0 + 2, \ldots\}$ the new ordinates are clearly those of $f(t - a)$. However, if we try to calculate them by invoking the de Casteljau-type algorithm (right-sided subdivision), the computation gets more complex, since it involves an infinite summation.

2.3. Convergence issues

In this section, we summarize the most remarkable results regarding the convergence of Poisson series, which are basically those of Taylor series (Needham, 1997). Since these results are formulated in the complex plane, henceforth we consider complex functions $f(z)$.

The Taylor or Poisson series (around the origin $z = 0$) of $f(z)$ converges to $f(z)$ within a certain circle C_r centred at $z = 0$, whose radius r is called *radius of convergence*, and diverges outside C_r. In principle, nothing can be said about the convergence on the rim of C_r. This *circle of convergence* C_r is the largest circle centred at the origin we can draw so that its interior contains only points where $f(z)$ is *analytic*. Those points z^* where $f(z)$ is not analytical are called *singular points* (or *singularities*). Hence, the circle C_r is the largest circle whose interior does not contain singularities z^* (Fig. 3(a)). Given a real interval $[0, d]$, we can guarantee convergence on this interval if $r > d$, that is, the circle C_d encloses no singularity.

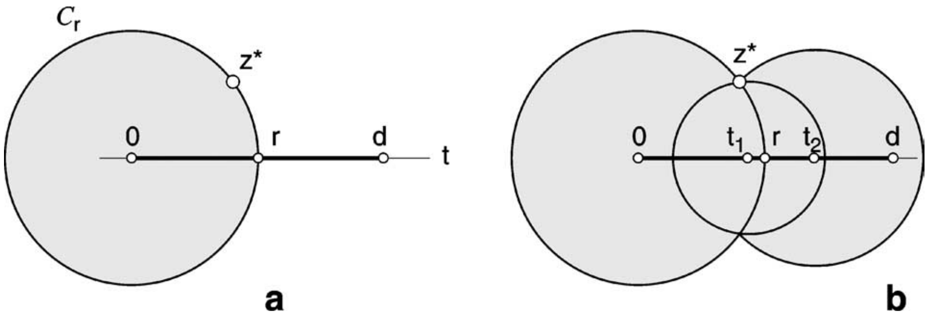

Fig. 3. (a) Circle C_r of convergence in the complex plane for a Taylor or Poisson series. (b) Avoiding a singularity z^* via analytic extension.

To avoid a singularity z^* not lying on the real axis, Morin and Goldman (2001a) employ analytical continuation to generate a sequence of convergent Poisson series, whose circles of convergence circumvent z^* (Fig. 3(b)). Suppose we want a convergent approximation on an interval $[0, d]$, $r < d$, so that a Poisson series centred at $t = 0$ cannot be employed beyond $t = r$. Via right-sided subdivision, we compute a new series centred at $t_1 < r$, hence enjoying convergence beyond $t = r$. This strategy can be employed again to further extend the interval of convergence or to avoid additional singularities.

We must remark an advantage of Poisson series over Taylor series for the case of entire functions (i.e., with $r = \infty$) and such that $\lim_{t\to\infty} f(t) = 0$. Then, the Poisson series converges uniformly over $[0, \infty)$, whereas the convergence of Taylor series is uniform only on finite intervals.

2.4. Truncated Poisson series and polygon

A first constraint in practical applications is that we cannot store an infinite series. Therefore, we must truncate expansion (3) at a certain kth term and get an approximation:

$$P_k(f;t) = \sum_{i=0}^{k} q_i b_i(\rho t), \tag{4}$$

which varies for different values ρ. As for Taylor series, the approximation $P_k(f;t)$ has contact of the kth order with $f(t)$ at $t = 0$. In contrast, a truncated Poisson series is not polynomial and hence not expressible in the Bernstein–Bézier standard. Therefore, it cannot be incorporated into commercial CAD programs. In fact, a truncated series cannot represent exactly *any* polynomial, except the null function.

Morin and Goldman (2000a) get around this drawback in a simple manner, by employing a truncated Poisson control polygon of points $\mathbf{q}_i$, $0 \leqslant i \leqslant k$, rather than a truncated series (4). To approximate $f(t)$ over an interval $t \in [0, d]$ with a polygon of $k+1$ points, simply choose a scale factor $\rho = d/k$. However, new problems arise with a truncated polygon:

- Neither the diminishing variation, nor the convex hull hold any more. The function $f(t) = t^2$ (Fig. 2) provides a simple counterexample.
- It yields a piecewise linear approximation, hence only C^0.

- The approximation does not interpolate $f(t)$ at the right endpoint. Thus, the analytic continuation described in Section 2.2 leads to gaps when connecting several expansions.

3. s-power series: the two-point analogue of Taylor series

3.1. Two-point Hermite approximation and Hermitian splines

A well-known technique in numerical analysis (Stoer and Burlirsch, 1993) is two-point Hermite approximation. Given a function $f(t)$ over a domain $[t_0, t_1]$, its order-k Hermite approximation $H_k(f;t)$ is the unique degree-$(2k+1)$ polynomial that has contact of the kth order with $f(t)$ at the endpoints t_0, t_1, in other words, the polynomial that interpolates all the derivatives $\{f^{(i)}(t_0), f^{(i)}(t_1)\}_{i=0}^{k}$. Such order-$k$ Hermite approximations are very convenient for a piecewise representation. If we split the domain into several segments and piece together their corresponding order-k Hermite approximations, we get a C^k polynomial spline function, called *Hermitian spline* (Grisoni et al., 1999). This piecewise function admits a standard degree-$(2k+1)$ B-spline representation, where the internal knots have multiplicity $k+1$.

Once we have identified the Hermite approximation as an attractive option, we must take into account that the customary order-k cardinal Hermite polynomials are not a subset of those of higher order, thereby not leading to Hermite series. Formally, the standard Hermite representation does not enjoy *permanence*. The permanence property is highly desirable: if the approximation error surpasses a determined threshold, we simply add one more term in the series, without having to recompute the whole approximation. A Newton form (Goldman, 2003) of nodes $\{t_0, t_1, t_0, t_1, \ldots\}$ solves the problem of the standard Hermite form, by furnishing two-point expansions over $[t_0, t_1]$, instead of a Taylor series around a point. In addition, the expansion admits the efficient $\mathrm{O}(k)$ nested evaluation the Newton form enjoys. Note that permanence could be also achieved through more complex multiresolution schemes, in particular the HB-splines (Hermitian B-splines) developed by Grisoni et al. (1999).

3.2. s-power series definition

Given a function $f(t)$ over a general domain $t \in [t_0, t_1]$, first we rewrite it as $a(u) = f(t(u))$ over a unit domain $u \in [0, 1]$, via a change of variable:

$$t(u) = t_0(1-u) + t_1 u, \quad u \in [0,1],\ t \in [t_0, t_1]. \tag{5}$$

Now we take the Newton representation of nodes $\{0, 1, 0, 1, 0, \ldots\}$, that is, where the points to interpolate have abscissas coalescing to the endpoints, and rearrange it to resemble a power series:

$$a(u) = \sum_{k=0}^{\infty} a_k(u) s^k, \quad s = (1-u)u,\ a_k(u) = (1-u)a_k^0 + u a_k^1. \tag{6}$$

The expansion over $u \in [0, 1]$ is called *s-power series* (Sánchez-Reyes, 2000), because it is simply a power series of symmetric parameter $s = (1-u)u$, whose coefficients $a_k(u)$ are linear functions of u, rather than constants. Note that $a_k(u)$ is expressed in Bernstein form, i.e., the pair a_k^0, a_k^1 denotes its Bézier ordinates (Fig. 4(a)).

The function $s^k(u)$ (6) is the central (kth) scaled Bernstein polynomial $\widehat{B}_k^{2k}(u)$:

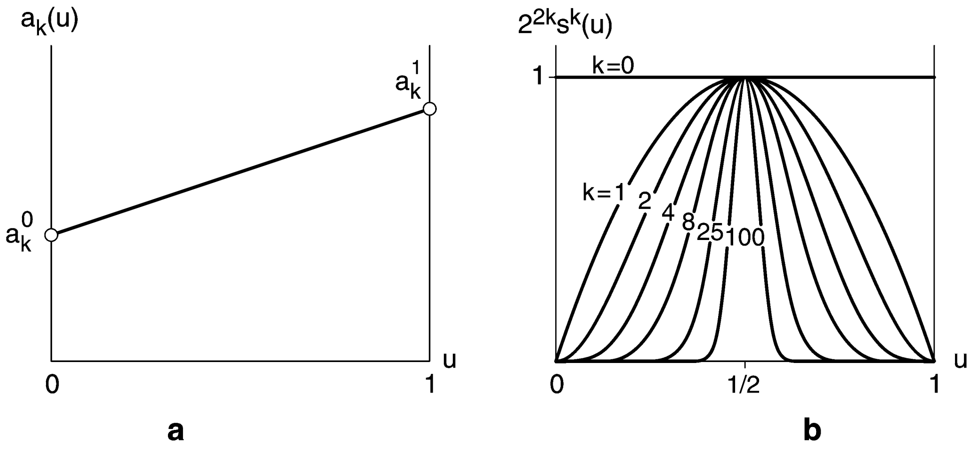

Fig. 4. Linear function $a_k(u)$ and symmetric function $s^k(u)$ in the kth term of a s-power series.

$$s^k = \widehat{B}_k^{2k}(u) = (1-u)^k u^k.$$

As shown in Fig. 4(b), this bell-shaped function displays the following properties:

- Degeneration to a scaled impulse as $k \to \infty$.
- Symmetry with respect to the midpoint $u = 1/2$.
- Unimodality, attaining the maximum 2^{-2k} at the midpoint.
- Contact of order $k-1$ with the horizontal axis, since it has a k-fold zero at $u = 0$, $u = 1$.

Given a function $a(u)$, analytic in a certain region of the complex plane, it can be expressed uniquely as a convergent s-power series (6). Moreover, the order-k Hermite interpolant $H_k(a; u)$ results from truncation at the kth term:

$$H_k(a; u) = \sum_{i=0}^{k} a_i(u) s^i. \tag{7}$$

Recall that this is the unique polynomial of degree $n = 2k+1$ that reproduces the derivatives $\{a^{(i)}(u)\}_{i=0}^{k}$ at the endpoints $u = 0$, $u = 1$.

The basis functions in expansion (6) are central pairs $\widehat{B}_k^{2k+1}$, $\widehat{B}_{k+1}^{2k+1}$ of scaled Bernstein polynomials (without binomial coefficients):

$$\widehat{B}_k^{2k+1} = (1-u)^{k+1} u^k, \quad \widehat{B}_{k+1}^{2k+1} = (1-u)^k u^{k+1}, \quad k = 0, 1, 2, \ldots.$$

Thus, the series admits a representation in terms of a certain control polygon (Sánchez-Reyes, 1997). Just compute the Bézier form of successive order-k (degree $n = 2k+1$) Hermite interpolants (7), and take pairs of central k, $(k+1)$th nonparametric Bézier points, over abscissas k/n, $(k+1)/n$. This control polygon satisfies the convex hull property. It also inherits the permanence property: a new kth term in the expansion refines the polygon by adding a pair of new central points, preserving the previous ones. However, for a convergent series the corresponding polygon converges to the function, as $k \to \infty$, only at the midpoint $u = 1/2$.

3.3. Representing and computing the coefficients a_k^0, a_k^1

Each “coefficient” $a_k(u)$ in a s-power series (6) is a linear function expressed in Bernstein form, thereby suggesting that a natural representation of $a_k(u)$ would be the pair of Bézier ordinates:

$$a_k(u) \to \mathbf{a}_k = \{a_k^0, a_k^1\}. \tag{8}$$

Therefore, representing a s-power series reduces, in essence, to representing a Taylor series where each coefficient is the duple (8), or to representing two Taylor series. Note that s-power series inherit the symmetry of the Bernstein representation, so that the series for $\tilde{a}(u) = a(1-u)$, denoted with a tilde, has coefficients $\tilde{\mathbf{a}}_k$ generated by swapping a_k^0, a_k^1.

To obtain explicit expressions of $\mathbf{a}_k$ (8) in terms of derivatives at the endpoints, simply apply divided differences (Goldman, 2003) in a triangular scheme. Given the Taylor series for the functions $a(u)$, $\tilde{a}(u) = a(1-u)$:

$$\begin{aligned} a(u) &= \sum_{i=0}^{\infty} c_i u^i, \quad c_i = \frac{1}{i!} a^{(i)}(0), \\ \tilde{a}(u) &= \sum_{i=0}^{\infty} \tilde{c}_i u^i, \quad \tilde{c}_i = \frac{1}{i!} \tilde{a}^{(i)}(0), \end{aligned} \tag{9}$$

the kth coefficient $\mathbf{a}_k$ is expressed as the linear combination:

$$\begin{aligned} \mathbf{a}_k &= \sum_{i=0}^{k} c_i \mathbf{h}_{i,k} + \tilde{c}_i \tilde{\mathbf{h}}_{i,k}, \\ \mathbf{h}_{i,k} &= \{1, 0\}, \quad i = k, \\ \mathbf{h}_{i,k} &= \left\{ \binom{2k-i-1}{k-i}, -\binom{2k-i-1}{k-i-1} \right\}, \quad i < k. \end{aligned} \tag{10}$$

This formula is a simplified version of that derived by López and Temme (2002), via complex contour integration, or by Fine (1961).

Next, we show that the pairs $\mathbf{h}_{i,k}$ of binomial numbers (10) multiplying the Taylor coefficients c_i (9) convey a clear meaning. First, we rewrite $a(u)$ as a linear combination of cardinal functions:

$$a(u) = \sum_{i=0}^{\infty} c_i h_i(u) + \tilde{c}_i \tilde{h}_i(u), \quad \tilde{h}_i(u) = h_i(1-u), \tag{11}$$

where the cardinal function $h_i(u)$ has derivatives that agree with those of the monomial u^i at $u = 0$, and vanish at $u = 1$. Comparing (10), (11) the pairs $\mathbf{h}_{i,k}$ are identified as the kth s-power duple (8) of the ith cardinal function $h_i(u)$, assuming $\mathbf{h}_{i,k} = \{0, 0\}$ for $i > k$. Fig. 5 plots these cardinal functions $h_i(u)$, for $i = 0, 1$, and their the successive order-k Hermite interpolants. Observe that $h_i(u)$ is expressed, over the real axis, in terms of a Heaviside step function $h_0(u)$ as:

$$h_i(u) = u^i h_0(u), \quad h_0(u) = \begin{cases} 1, & u < \frac{1}{2}, \\ \frac{1}{2}, & u = \frac{1}{2}, \\ 0, & u > \frac{1}{2}. \end{cases}$$

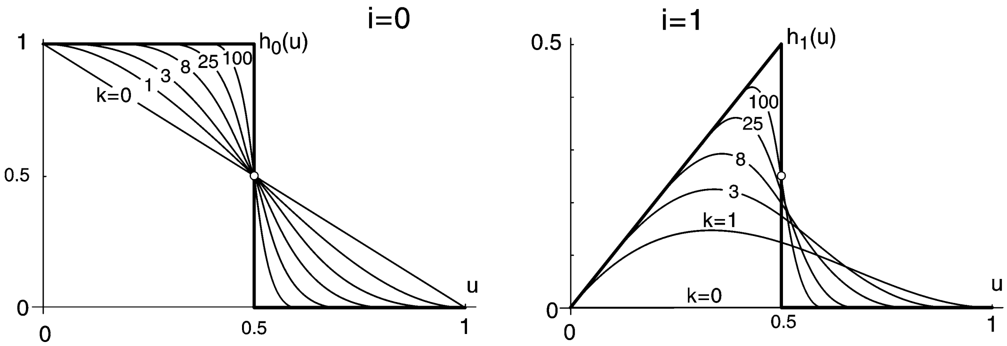

Fig. 5. Successive order-k Hermite expansions for the cardinal functions $h_i(u)$, $i = 0, 1$, $u \in [0, 1]$.

The pair $\mathbf{a}_k$ can also be computed without knowledge of the derivatives, by combining basic s-power series (Sánchez-Reyes, 2000). The resulting algorithms are akin to those for Taylor series (Knuth, 1998), as the basic idea in the definition (6) is to rearrange the Newton form, so that it can be manipulated like a standard power series.

3.4. *Lemniscates of convergence*

In this section, we summarize the results regarding the convergence of s-power series (Sánchez-Reyes, 2000). Such results derive from applying the classical convergence theorems for interpolatory processes to the particular case of symmetric two-point Hermite interpolation. Formal proofs can be found in the treatise (Davis, 1975).

The convergence of complex s-power series is formulated in the complex plane, as for Taylor series. Since a s-power series is a power series in $s(z) = (1 - z)z$, simply replace as areas of convergence the circles $z = r$ with curves:

$$\left|s(z)\right| = r^2, \quad s(z) = (1 - z)z. \tag{12}$$

Such curves are known as *Cassinian curves* (Needham, 1997). They are the locus of points z such that the product of their distances to the foci $(z = 0, z = 1)$ is a constant r^2. Thus, the s-power series of a function $a(z)$ converges uniformly to $a(z)$ within the largest Cassinian curve of foci $z = 0$, $z = 1$ we can draw so that it does not contain any singularity, and diverges outside. Fig. 6 shows that a varying r generates confocal curves (12) belonging to 3 different families:

- $0 < r < 1/2$: two *Ovals of Cassini*, one surrounding $z = 0$ and the other $z = 1$.
- $r = 1/2$: *Lemniscate L of Bernoulli*, an ∞-shaped figure with a double point at $u = 1/2$.
- $r > 1/2$: a closed contour containing both foci.

In most applications, we are interested only in the convergence over the real unit interval $[0, 1]$ of definition. To guarantee a convergent behaviour on $[0, 1]$, this interval must be completely contained in the complex area of convergence. As only Cassinian curves with $r > 1/2$ embrace $[0, 1]$, the lemniscate L of Bernoulli must not contain singularities. However, if L contains a singularity z^*, we can always circumvent it through subdivision. Fig. 7 illustrates this strategy for a singularity z^* not lying on the real segment $[0, 1]$. If we subdivide it, and compute the s-power series for each segment, their

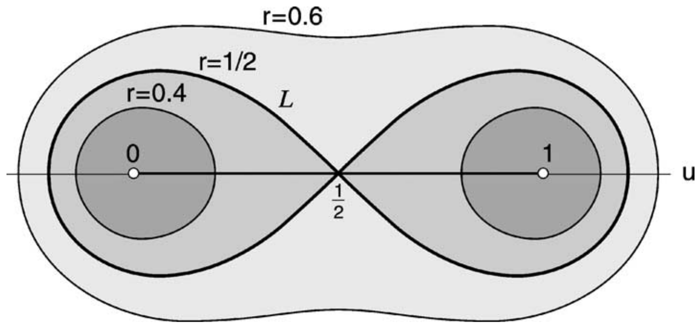

Fig. 6. Confocal areas of convergence in the complex plane for s-power series.

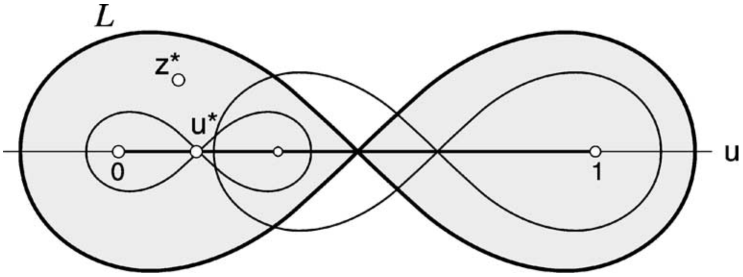

Fig. 7. Avoiding a singularity z^* via subdivision in s-power series.

new limiting lemniscates L are smaller. Clearly, after successive subdivisions any singularity can be avoided, and hence we achieve convergent s-power series for each segment. As already commented, truncating these s-power series and piecing them together furnishes a smooth spline. In case of a real singularity $u^* \in (0,1)$, we split $[0,1]$ so that u^* lies just in the midpoint of one of the resulting intervals. As u^* coincides with the double point of the corresponding lemniscate, convergence is guaranteed except at u^*.

We stress that, in this context, *subdivision* is tantamount to splitting the domain of the given function $f(t)$ into several pieces, and for each piece computing a truncated series $H_k(a;u)$ (7). It does not mean subdividing the polynomial approximation furnished by a truncated series, so no de Casteljau-like algorithm is employed.

4. Comparative examples

In this section, we justify first why better convergence can be expected for s-power series than for Poisson series. Second, we take several examples from Morin and Colgman (2000a, 2000b, 2001a) and plot the approximations over a given interval $t \in [0,d]$ furnished by s-power series and Poisson series and polygons. For an unbiased comparison, we refer to the number of terms in both approximations, namely the degree $n = 2k + 1$ for s-series (7), and the index k (4) in the Poisson case.

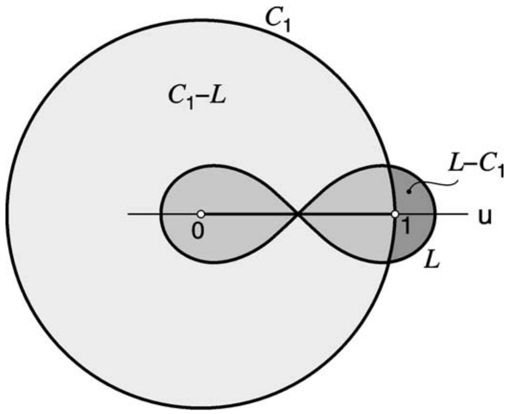

Fig. 8. Comparing the limiting areas of convergence for Poisson and s-power series.

4.1. Why s-power series display better convergence than Poisson series

Suppose we want to construct a series over the real segment $u \in [0, 1]$. Fig. 8 displays together the limiting areas, a circle C_1 for Poisson and a lemniscate of Bernoulli L for s-power series, that determine the convergence over this segment. If the area contains no singularity, then the corresponding series enjoys convergence. The lemniscate L hugs this segment closer than the circle C_1, so that the chance to contain a singularity z^* is lower. A singularity z^* in the wide region $C_1 - L$ implies a divergent Poisson series, whereas the s-power series enjoys still convergence. Only a singularity falling in the small region $L - C_1$ results in a convergent Poisson series and a divergent s-power series. We conclude that, in general, s-power series enjoy better convergence than Poisson series.

4.2. Logarithmic function $f(t) = \log(2 - 2t + t^2)$

In a first example, we approximated on $t \in [0, 3]$ the function $\log(2 - 2t + t^2)$, which has two complex singularities $z^* = 1 \pm \mathrm{i}$ (Fig. 9(a)). The s-power series is convergent over $[0, 3]$, since both singularities lie outside the limiting lemniscate. This convergent behaviour is shown in Fig. 9(b) for successive s-power approximations (degrees $n = 3, 5, 7$). In contrast, as the Poisson series has a radius of convergence $r = \sqrt{2}$, it is not convergent beyond $t = r$, and hence the convergence of the control polygon is neither guaranteed (Fig. 9(c)). Fig. 9(d) compares the error $\varepsilon(t)$ for both approximations, showing the higher accuracy of s-power series.

4.3. Rational function $f(t) = 1/(1 + t)$

Another example is provided by the rational function $(1 + t)^{-1}$ over $t \in [0, 2]$, which has a real singularity $t^* = -1$ (Fig. 10(a)). Such a singularity lies outside the limiting lemniscate, so the s-power series (Fig. 10(b)) is convergent. In contrast, the Poisson series is divergent beyond $r = 1$ (Fig. 10(c)), and the convergence of the Poisson polygons is not guaranteed beyond this point. Plotting the error $\varepsilon(t)$ for both approximations (Fig. 10(d)) shows again that, to achieve similar accuracy, more terms are needed using a Poisson polygon.

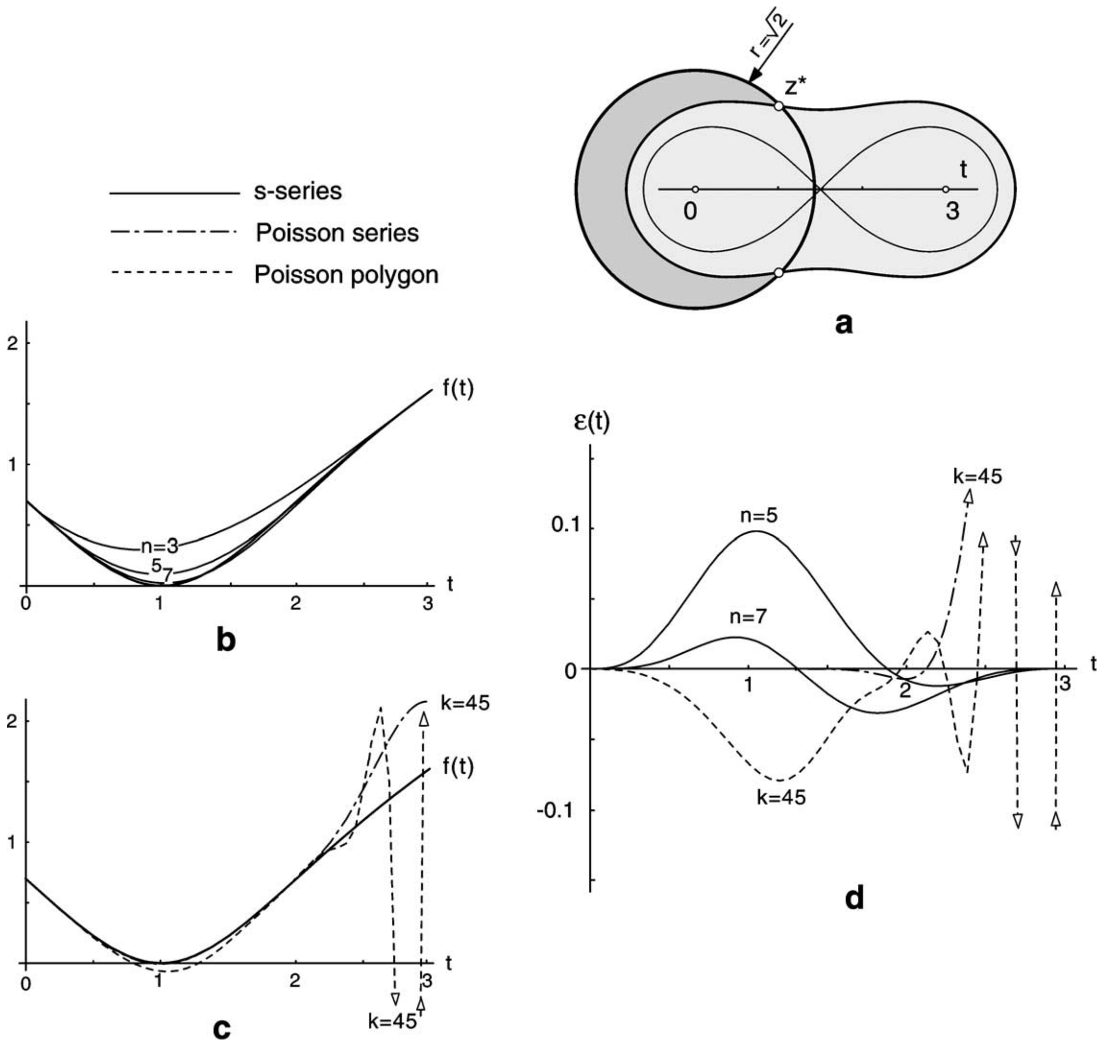

Fig. 9. $f(t) = \log[2 - 2t + t^2]$. (a) Areas of convergence for Poisson and s-power series. (b) s-power approximation (degrees $n = 3, 5, 7$). (c) Truncated Poisson polygon and series ($k = 45$). (d) Errors $\varepsilon(t)$.

4.4. Archimedean spiral

We computed several approximations to the Archimedean spiral $\mathbf{c}(t) = (t\cos t, t\sin t)$ on $t \in [0, 4\pi]$. The component functions are entire, and hence convergence is guaranteed for both the s-power and Poisson approximations. Fig. 11(a) plots the s-power series (degree $n = 7, 9$), along with the Poisson control polygon ($k = 400, 800$). The quality of the approximations is measured using a Pythagorean error $\varepsilon(t)$, i.e., distance between the exact curve $\mathbf{c}(t)$ and the approximation for a value t. For s-power series, we subdivided the initial domain at the midpoint $t = 2\pi$ to subdue the maximum error. As shown in Fig. 11(b), $\varepsilon(t)$ vanishes at the endpoints and is bell-shaped. Notice that, to achieve an acceptable

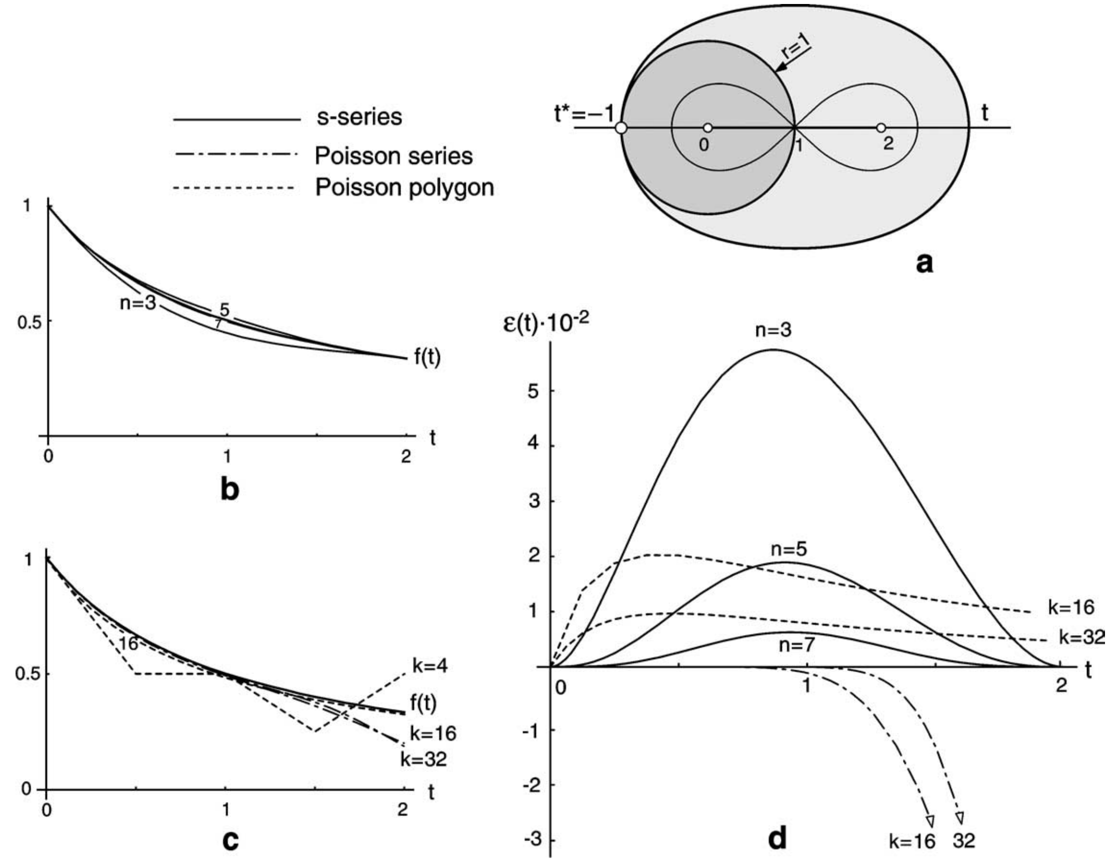

Fig. 10. $f(t) = (1+t)^{-1}$. (a) Singularity t^* of $f(t)$ and areas of convergence. (b) s-power approximation (degree $n = 3, 5, 7$). (c) Truncated Poisson series ($k = 16, 32$) and Poisson polygons ($k = 4, 16, 32$). (d) Error $\varepsilon(t)$.

maximum error, a degree-9 s-power series suffices, whereas both the truncated Poisson polygon and series for $k = 800$ are still visually distinguishable from the exact curve.

4.5. Unit semicircle

In Fig. 12 we consider a trigonometrically parameterized semi-circle $\mathbf{c}(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. As for the spiral, the components are entire functions, and hence convergence is guaranteed for both approximations. However, s-power series furnish a result of higher quality, with a more evenly distributed error. This example illustrates another advantage of Hermite interpolation (Sánchez-Reyes and Chacón, 2003), namely that it tries to preserve the original arc-length parameterization, as shown by the parametric speed $\sigma(t) = |\mathrm{d}\mathbf{c}(t)/\mathrm{d}t|$.

4.6. Bivariate surfaces

Finally, in Fig. 13 we approximated the bivariate function $f(x, y) = \sin x \sin y$ over the domain $(x, y) \in [0, 2\pi] \times [0, 2\pi]$. Clearly, both s-power and Poisson series enjoy convergence, but a trun-

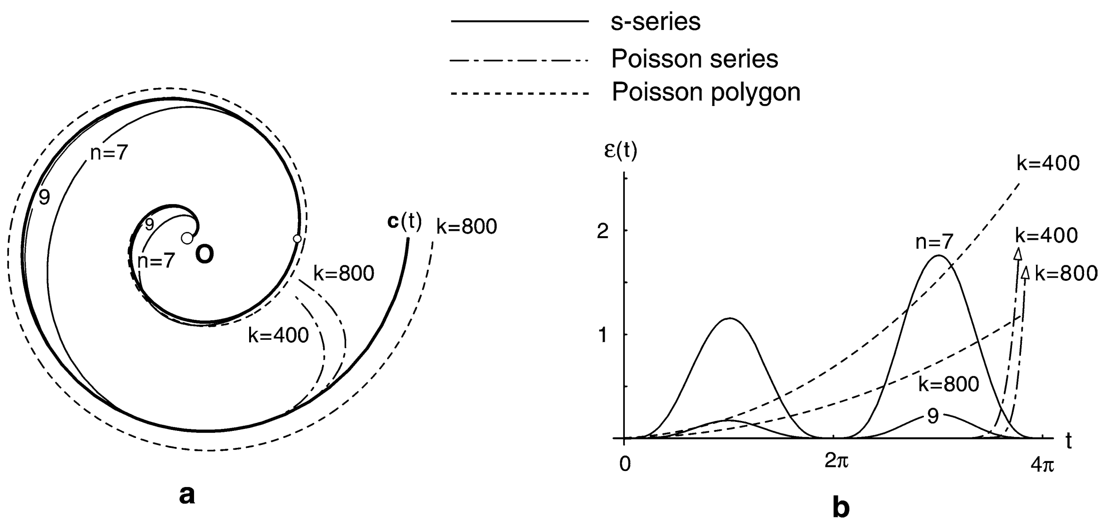

Fig. 11. (a) Approximations: s-power (degrees $n = 7, 9$) and Poisson ($k = 400, 800$). (b) Error $\varepsilon(t)$.

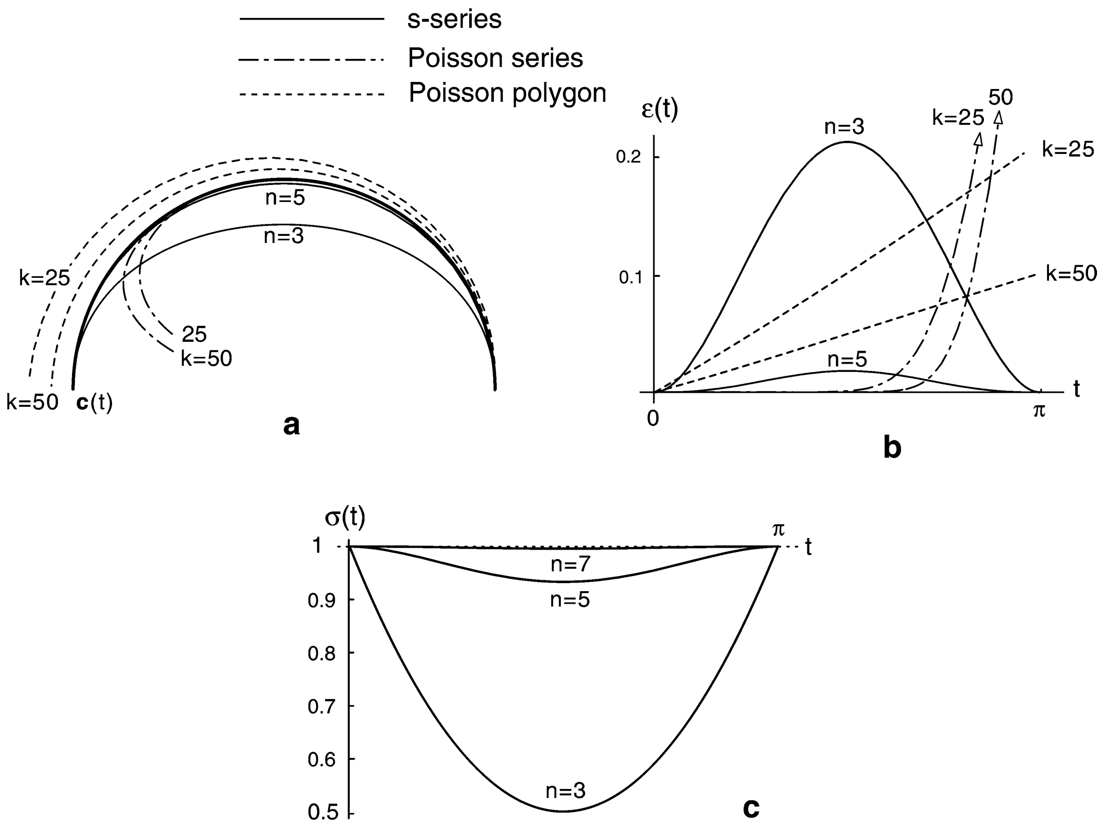

Fig. 12. Semicircle. (a) Approximations: s-power (degree $n = 3, 5$) and Poisson ($k = 25, 50$). (b) Errors $\varepsilon(t)$. (c) Parametric speed $\sigma(t)$ for the s-power approximations.

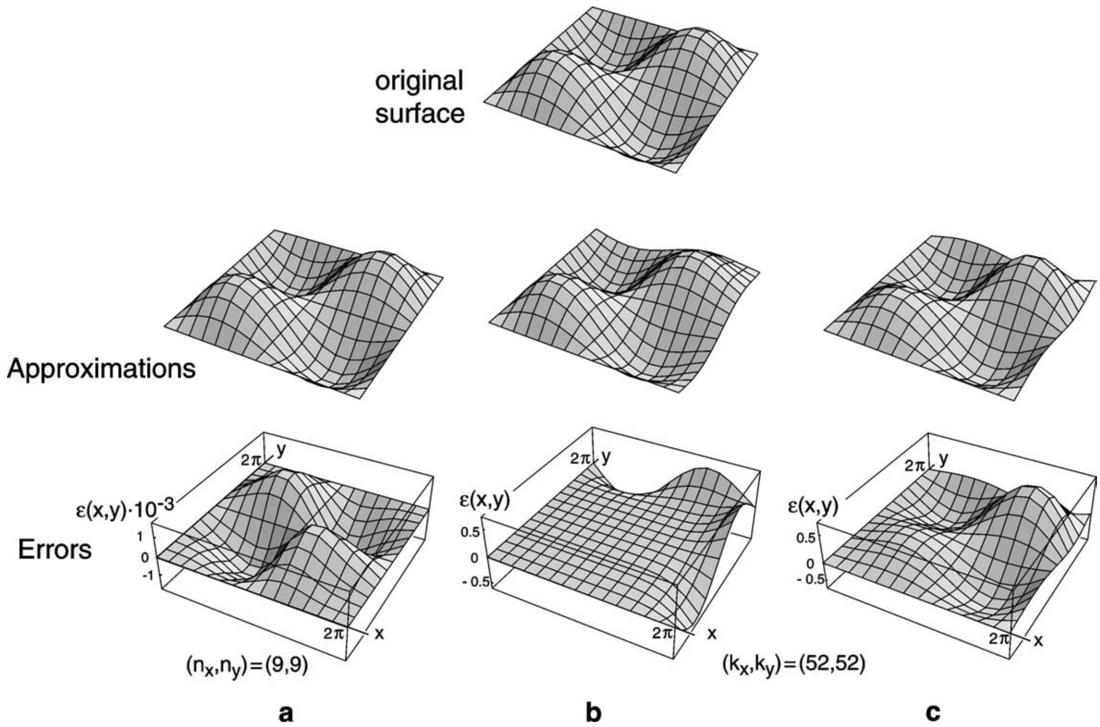

Fig. 13. $f(x, y) = \sin x \sin y$. Approximations and corresponding errors $\varepsilon(x, y)$: s-power (a), degree $(n_x, n_y) = (9, 9)$; Poisson series (b) and mesh (c), $(k_x, k_y) = (52, 52)$.

cated s-power series displays better accuracy. The same applies for the approximations of a catenoid, the surface of revolution obtained by revolving a catenary $r(z) = \cosh z$ around the vertical axis. Fig. 14 compares the approximations for one half of the catenoid, corresponding to the domain $(z, \theta) \in [-2, 2] \times [0, \pi]$.

5. Conclusions

Poisson and s-power series provide different options for describing arbitrary analytic functions in CAGD. Poisson functions are the elegant generalization of nonparametric Bézier curves over a semi-infinite domain, endowed with an infinite control polygon that displays the advantageous properties of the Bézier scheme. However, in practical CAD applications geometric entities are usually defined over finite intervals, and we cannot deal with infinite control polygons or infinite series. Thus, we must perform some kind of truncation, which leads to the following problems in the Poisson case:

- A truncated Poisson series results in a non-polynomial approximation, which cannot be incorporated into existing CAD systems, based on the Bernstein–Bézier standard.
- If we employ as approximation a truncated Poisson polygon, it is only C^0.
- Concatenating several approximations to avoid a singularity yields discontinuities at the junctures.

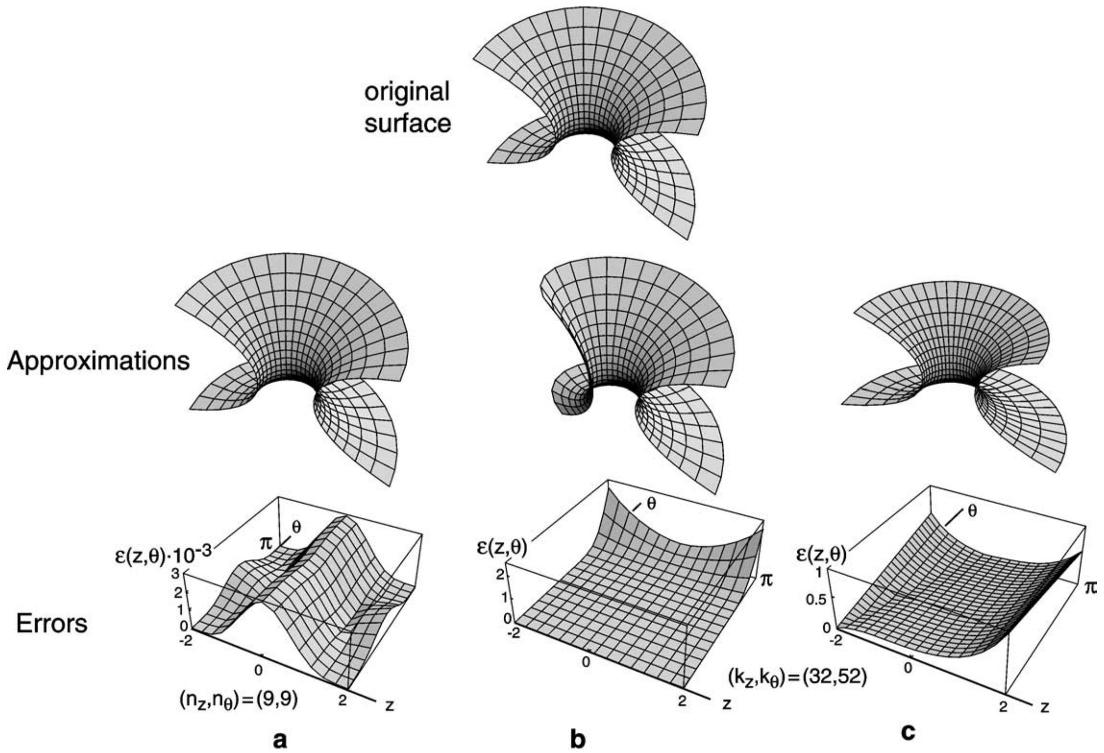

Fig. 14. Semi-catenoid. Approximations and corresponding errors $\varepsilon(z,\theta)$: s-power (a), degree $(n_z, n_\theta) = (9, 9)$; Poisson series (b) and mesh (c), $(k_z, k_\theta) = (32, 52)$.

- Truncations do not enjoy permanence: to increase the precision of an approximation over a given interval, a new polygon replacing the old one must be computed.

Two-point Hermite expansions (s-power series) lack the blossoming and Bézier-like tools of Poisson functions, including the representation in terms of a converging control polygon. Nevertheless, for approximating analytic functions over finite intervals, they seem a practical alternative to Poisson or Taylor series:

- They are polynomial, and hence expressible in the Bernstein–Bézier standard.
- By piecing approximations of order k (degree $2k + 1$) we obtain a Hermitian spline, whose segments join smoothly with C^k continuity.
- They display better convergence and accuracy than Poisson series, and any singularity is easily avoided via subdivision.
- They enjoy permanence: to increase precision, just add more terms.

These Hermite expansions are called s-power series, because over a domain $u \in [0, 1]$, they are nothing else than a power series of parameter $s = (1 - u)u$, and coefficients that are linear functions of u. Hence, they can be handled much like standard power series.

Acknowledgement

This work is supported by the Spanish Ministerio de Ciencia y Tecnología, under research grant DPI2003-02598.

References

Davis, P.J., 1975. Interpolation and Approximation. Dover, New York.
DeRose, T.D., 1988. Composing Bézier simplexes. ACM Trans. Graph. 7 (3), 198–221.
Farin, G., 2001. Curves and Surfaces for Computer Aided Geometric Design, fifth ed. Morgan Kaufmann, San Mateo, CA.
Fine, M., 1961. A note on two-point Taylor's series for solving ordinary differential equations. J. Aerospace Sci. 28, 671–672.
Goldman, R.N., 2003. Pyramid Algorithms. A Dynamic Approach to Curves and Surfaces for Geometric Modelling. Morgan Kaufmann, San Mateo, CA.
Grisoni, L., Blanc, C., Schlick, C., 1999. Hermitian B-splines. Computer Graphics Forum 18 (4), 237–248.
Farouki, R.T., 2000. Convergent inversion approximations for polynomials in Bernstein form. Computer Aided Geometric Design 17 (2), 179–196.
Hermann, T., 2002. Degree elevation for generalized Poisson functions. Computer Aided Geometric Design 19 (1), 67–70.
Knuth, D.E., 1998. The Art of Computer Programming, Seminumerical Algorithms, vol. 2, third ed. Addison-Wesley, Reading, MA.
Lawrence, J.D., 1972. A Catalog of Special Plane Curves. Dover, New York.
Lee, B.-G., Park, Y., Yoo, J., 2002. Application of Legendre–Bernstein basis transformations to degree elevation and degree reduction. Computer Aided Geometric Design 19 (9), 709–718.
Li, Y.-M., Hsu, V.Y., 1998. Curve offsetting based on Legendre series. Computer Aided Geometric Design 15 (7), 711–720.
López, J., Temme, N.M., 2002. Two-point Taylor expansions of analytic functions. Stud. Appl. Math. 109 (4), 297–311.
Morin, G., Goldman, R., 2000a. A subdivision scheme for Poisson curves and surfaces. Computer Aided Geometric Design 17 (9), 813–833.
Morin, G., Goldman, R., 2000b. Poisson approximation. In: Geometric Modelling and Processing 2000, April 10–12, 2000, Hong Kong, IEEE Computer Society, pp. 141–149.
Morin, G., Goldman, R., 2001a. Trimming analytic functions using right sided Poisson subdivision. Computer-Aided Design 33 (11), 813–824.
Morin, G., Goldman, R., 2001b. The analytic blossom. In: Lyche, T., Schumaker, L.L. (Eds.), Mathematical Methods for Curves and Surfaces, Oslo, 2000. Vanderbilt University Press, Nashville, pp. 325–346.
Needham, T., 1997. Visual Complex Analysis. Clarendon Press, Oxford.
Peña, J.M., 1999. Shape Preserving Representations in Computer-Aided Geometric Design. Nova Science, New York.
Piegl, L., Tiller, W., 1997. The NURBS Book, second ed. Springer, Berlin.
Sánchez-Reyes, J., 1997. The symmetric analogue of the polynomial power basis. ACM Trans. Graph. 16 (3), 319–357.
Sánchez-Reyes, J., 2000. Applications of the s-power basis in geometry processing. ACM Trans. Graph. 19 (1), 27–55.
Sánchez-Reyes, J., Chacón, J.M., 2003. Polynomial approximation to clothoids via s-power series. Computer-Aided Design 35 (14), 1305–1313.
Stoer, J., Burlirsch, R., 1993. Introduction to Numerical Analysis. Springer, Berlin.