Constructing Integers and Rational Numbers:

a good use of equivalence relations

In another handout, we saw how one could construct/create the natural numbers

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

using sets. The construction "worked" since what we got out satisfied the *Peano Axioms*. From this, one can define addition and multiplication and get the familiar properties we all love. Once we have $\mathbb{N}$ together with addition and multiplication, getting the integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$ is a great exercise in the use of equivalence relations.

First let's look at (look for?) the integers. What is "missing" from the natural numbers? The answer is "additive inverses." For example, what should one add to 7 to get the additive identity 0? If you said "negative seven," you're getting ahead of the game since all we have for the moment is the natural numbers, and -7 is not one of those. In fact, addition in $\mathbb{N}$ is the whole *raison d'etre* for $\mathbb{Z}$, (and similarly multiplication in $\mathbb{Z}$ points to the need for $\mathbb{Q}$.) Specifically, we want $\mathbb{Z}$ (i.e., negative numbers) so that everything has an additive inverse.

Say two ordered pairs (x, y) and (n, m) of natural numbers are *equivalent* if $x + m = y + n$. It's important that we don't (more simply?) say $x - y = n - m$. Why? Because $x - y$ doesn't make sense all the time. After all, if the only thing we have to work with at present is $\mathbb{N}$, then there is no "$5 - 8$", for example. We need to show that this is an equivalence relation.

Since $x + y = y + x$ by commutativity of addition in $\mathbb{N}$, (x, y) is equivalent to (x, y). That's the reflexive property. If $x + m = y + n$, then by commutativity of addition in $\mathbb{N}$, $n + y = m + x$. In other words, if (x, y) is equivalent to (n, m), then (n, m) is equivalent to (x, y). That's the symmetric property. Finally, suppose (x, y) is equivalent to (n, m), which is in turn equivalent to (s, t). Then $x + m = y + n$ and $n + t = m + s$. Thus

$$\begin{aligned}
(x + t) + m &= x + (t + m) \\
&= x + (m + t) \\
&= (x + m) + t \\
&= (y + n) + t \\
&= y + (n + t) \\
&= y + (m + s) \\
&= y + (s + m) \\
&= (y + s) + m.
\end{aligned}$$

Now since $(x + t) + m = (y + s) + m$, by the properties of addition in $\mathbb{N}$ we have $x + t = y + s$, i.e., (x, y) is equivalent to (s, t) and the transitive property is proven.

The integers $\mathbb{Z}$ are then defined to be the set of *equivalence classes* of ordered pairs of natural numbers. Think of the equivalence class containing (a, b) as representing $a - b$. We "define" addition and multiplication as follows:

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$

$$[(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)],$$

where $[(a, b)]$ denotes the equivalence class containing (a, b). I put the quotes on "define" here since, technically speaking, one has to verify that this makes sense. You see, these definitions use particular elements of an equivalence class to represent the entire equivalence class, and that could conceivably lead to difficulty. (We had the exact same issue when "defining" addition and multiplication of congruence classes in chapter 1.)

As before, one has to check that replacing (a, b) above by (x, y) whenever (x, y) is equivalent to (a, b) doesn't change the sum or product. Luckily that isn't difficult, and it turns out that $\mathbb{Z}$ satisfies all of the axioms we stated on the first day of class.

What happened to $\mathbb{N}$ in all this? After all, the natural numbers are supposed to be a subset of the integers, right? Actually, it isn't hard to get around this. Given a natural number n, we can view it as an element of $\mathbb{Z}$ by equating n with $[(n, 0)]$. In fact, we can list all of $\mathbb{Z}$ in a similar manner:

$$\ldots [(0, 3)], [(0, 2)], [(0, 1)], [(0, 0)], [(1, 0)], [(2, 0)], [(3, 0)], \ldots$$

Now that you've seen how to get $\mathbb{Z}$ from $\mathbb{N}$, getting the rational numbers $\mathbb{Q}$ from $\mathbb{Z}$ is very similar. We created $\mathbb{Z}$ in order to have "additive inverses" for all our numbers. We create $\mathbb{Q}$ in order to have "multiplicative inverses" for all non-zero numbers.

Say two ordered pairs $(a, b), (c, d)$ of integers with $bd \neq 0$ are equivalent if $ad = bc$. As with the equivalence defined on ordered pairs of natural numbers above, this is an equivalence relation. In fact, the same proofs we had above carry over exactly just by replacing $+=$ with $\times$. The rational numbers $\mathbb{Q}$ are defined to be the set of equivalence classes. Addition and multiplication are defined in the familiar way (think of $[(a, b)]$ as $\frac{a}{b}$):

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)] \qquad [(a, b)] \times [(c, d)] = [(ac, bd)].$$

Again, one must make sure this definition makes sense, i.e., doesn't depend on the particular element of the equivalence class used.

For an integer $a \in \mathbb{Z}$, we view it as an element of $\mathbb{Q}$ by equating it with $[(a, 1)]$. This makes $\mathbb{Z}$ a subset of $\mathbb{Q}$ (well, kind of).