Folding and Cutting Paper

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Abstract. We present an algorithm to find a flat folding of a piece of paper, so that one complete straight cut on the folding creates any desired plane graph of cuts. The folds are based on the straight skeleton, which lines up the desired edges by folding along various bisectors; and a collection of perpendiculars that make the crease pattern foldable. We prove that the crease pattern is flat foldable by demonstrating a family of folded states with the desired properties.

1 Introduction

Take a sheet of paper, fold it into some flat origami, and make one complete straight cut. What shapes can the unfolded pieces make? For example, Figure 1 shows how to cut out a five-pointed star in this way. You could imagine cutting out the silhouette of your favorite animal, object, or geometric shape.

The first published reference to this *fold-and-cut* idea that we are aware of is a Japanese book [22] by Kan Chu Sen from 1721. This book contains a variety of problems for testing mathematical intelligence [21]. One of the problems asks to fold a rectangular piece of paper flat and make one complete straight cut, so as to make a symmetric "zig-zag" polygon. The author gives a solution that consists of a sequence of simple folds, each of which folds along a line. See Figure 2.

Folding and cutting has also been used for a magic trick by Houdini, before he became a famous escape artist. In his 1922 book *Paper Magic* [11], he describes a method for making a five-pointed star, similar to the one in Figure 1. Another magician, Gerald Loe, studied this idea in some detail; his *Paper Capers* [19] describes how to cut out arrangements of various geometric objects, such as a circular chain of stars. Martin Gardner wrote about this problem in his famous

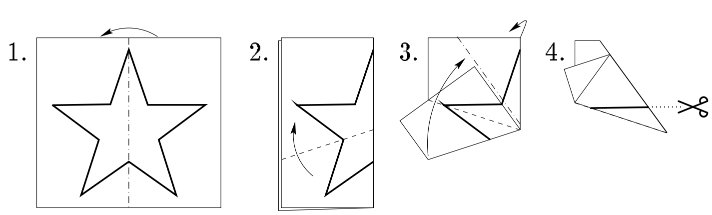

Fig. 1. *How to fold a square of paper so that one cut makes a five-pointed star.*

Title page | Problem statement | Solution (page 1) | Solution (page 2)

Fig. 2. A few pages from [22].

series in *Scientific American* [9]. He was particularly impressed with Loe's ability to cut out any desired letter of the alphabet.

Gardner [9] was the first to state cutting out complex polygons as an open problem. What are the limits of this fold-and-cut process? What polygonal shapes can be cut out?

In this paper, we prove that any collection of straight edges can be cut along, by a single straight cut after folding flat. This includes multiple disjoint, nested, and/or adjoining polygons, as well as floating line segments and points: a general plane graph. To solve this problem, we present an algorithm that computes the creases and the actual flat origami that lines up precisely the given plane graph. Cutting along this line hence achieves the desired result.

The rest of this paper is outlined as follows. Section 2 provides formal definitions of folds and cuts. Section 3 states our main theorem, and presents several interesting consequences. In Section 4, we specify the basic crease pattern for our solution. Section 5 concentrates on specifying the possible foldings, and proving the correctness of the algorithm. We conclude in Section 6.

2 Background

Origami mathematics is the study of the geometry and other properties of origami (paper folding). The area of origami mathematics is still in its infancy, having only been seriously studied for the past twenty years. Geretschläger [10] and Huzita and Scimemi [13] examined the geometric constructions possible with origami, and compared them to a ruler and compass. Bern and Hayes [4] showed that it is NP-hard to determine whether a crease pattern is flat foldable, as is computing a flat folding (overlap order) given a suitable direction of folds (mountain-valley assignment). Hull [12] and Kawasaki [15] focus on necessary and sufficient conditions for flat foldability of crease patterns with a single vertex, which are also necessary conditions for general crease patterns. Justin [14] examines necessary and sufficient conditions on overlap orders, resulting in a characterization of flat foldability for general crease patterns.

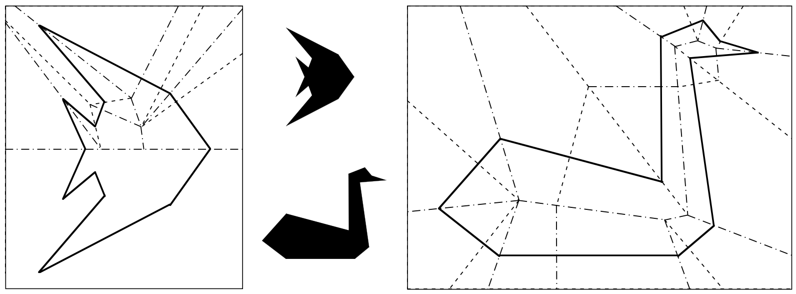

Fig. 3. *Minimal crease patterns for an angelfish and a swan. The cut graph is drawn thick, and valleys [mountains] are drawn dashed [dot-dashed]. For the angelfish, fold in half first.*

Lang has taken the most algorithmic approach. In [17], he describes an algorithm to construct "uniaxial" bases, which can then be folded into arbitrarily complex models. This solves a major problem in origami sekkei (technical folding). Lang's work is related to ours: essentially, a portion of his solution deals with the fold-and-cut problem when the shapes to cut are all convex polygons.

Two other papers study the fold-and-cut problem formally. Demaine and Demaine [5] solve the problem of folding a polygonal sheet of paper to map the paper's boundary to a line. This result is in one sense much weaker and in one sense stronger than the present result. It is weaker in that it solves the fold-and-cut problem only for convex polygons, where the folds exterior to the polygon do not interfere. It is stronger in that it describes the exact folding process, that is, the function that folds the piece of paper through time to the final folded state. This shows that the folding can be achieved while keeping the paper rigid (except at the creases), and allows animation of the folding process.

Inspired by preliminary versions of this work, Bern, Demaine, Eppstein, and Hayes [3] have proposed an alternative solution to the fold-and-cut problem using the idea of disk packing. That solution is more "local" than the one presented here, which exploits and demonstrates the global structure of the problem. The advantage of the disk-packing solution is that the number of folds is bounded in terms of the number of vertices and minimum feature size. The origamis presented here, on the other hand, have the advantages of being more natural and easier to fold in practice. Our techniques have also helped extend work in algorithmic origami design [17, 18].

The rest of this section defines the terminology used in this paper.

A *plane graph* is a planar graph with a fixed embedding such that every edge is straight and has positive length, and every pair of (closed) edges intersects only at a shared vertex. We allow edges to have zero, one, or two actual vertices, corresponding to infinite lines, half-infinite lines, and line segments, respectively.

A *crease pattern* is simply a plane graph. We will find it easier to consider folding an infinite plane, although the actual piece of paper will be a bounded subset of that. An *origami* or *folding* of a crease pattern [4, 5, 12, 17] is a contin-

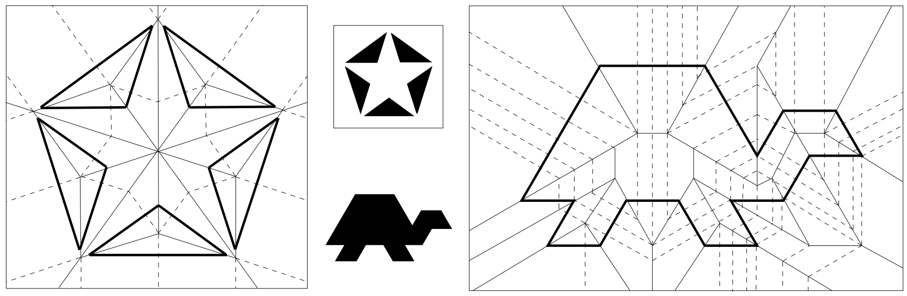

Fig. 4. *Full crease patterns for a fancy star and a turtle. The cut graph is drawn thick, the straight skeleton is drawn solid, and the perpendiculars are dashed.*

uous function from $\mathbf{R}^2$ to $\mathbf{R}^3$ with the following properties. First, the function must map every face of the crease pattern to a congruent copy in three dimensions. Second, the folding must not induce any crossings; one way to define this is to allow faces to be an infinitesimal distance apart (thereby defining their order), and enforce that the folding be a one-to-one function.

Note that a folding gives the folded state, not the process of how to get to the folded state.

A *flat origami* is an origami whose image lies on a plane. We can define the *mountain-valley assignment* of a flat origami by assigning either "mountain" or "valley" to each edge in the crease pattern, according to whether it was folded by angle π or angle $-\pi$, relative to the top side of the piece of paper.

2.1 Models of Cuts

This section specifies exactly what we mean by "making a cut." A (*complete*) *cut* is a line. The natural mathematical model for applying a cut C to some object O is to remove all the points along C from O, that is, take $O - C$. In particular, if the piece of paper is an open set (e.g., the entire plane, or a polygon without its boundary), then the resulting pieces will also be open sets. This model is due to Frederickson [8], and we call it a *mathematical cut*.

Mathematical cuts are best modeled in real life by a laser. In particular, they do not correspond precisely to cutting with scissors, the problematic case being a fold and a cut that coincide. As an alternative to the mathematical cut model described above, we define a *scissor cut* so that when a fold and a cut coincide, the points on the fold are not removed. See Figure 5.

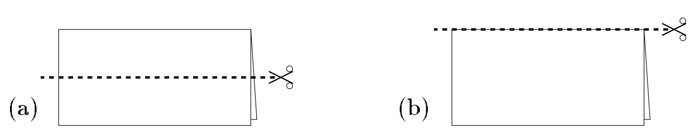

Fig. 5. *Cut (a) can be made by a scissor cut, whereas Cut (b) cannot.*

3 Results

We are now in the position to formally state the fold-and-cut problem. We are given a plane graph called the *cut graph*. We refer to vertices, edges, and faces of the cut graph as simply "cut vertices," "cut edges," and "cut faces." Each cut face is also given a *side* of "above" or "below," in what we call the *side assignment*. The problem is to find a flat folding of the paper and a line l, called the *cut line*, such that the intersection of the folding with l is exactly the (folded) cut graph. The folding must place cut faces above or below the cut line l according to the given side assignment.

The most general result we could hope for is the following conjecture.

Conjecture 1. Given a cut graph and any side assignment, there exists a flat folding of the plane that maps the cut graph and nothing else to a common line, and appropriately maps cut faces above or below this line.

We will just fall short of proving this with the following theorem. The idea of linear and circular corridors will be described in Section 4.4.

Theorem 1. *Given a cut graph and a side assignment, such that either (1) the cut graph induces no circular corridors, or (2) the side assignment is constant (that is, it assigns all cut faces to the same side), then there exists a flat folding of the plane that maps the cut edges and nothing else to a common line, and appropriately maps faces above or below this line.*

Precisely what this theorem says in terms of the original fold-and-cut application depends on the kind of cuts allowed. For mathematical cuts, we have the desired result that any cut graph can be achieved, by choosing the side assignment that puts every face above the cut line. If only scissor cuts are allowed, then we need a side assignment with the property that every cut edge is incident to two faces assigned to opposite sides (see Figure 5). In other words, we want a face 2-coloring of the cut graph, where the colors correspond to "above" and "below." This is equivalent to the cut graph being even, that is, having all vertices of even degree [16].

The following summarizes our results for both kinds of cuts.

Corollary 1. *Given any cut graph, there exists a flat folding of the plane and a line on this folding such that mathematically cutting along the line removes precisely the (folded) cut graph. If only scissor cuts are allowed, this result holds for even cut graphs that induce no circular corridors.*

If Conjecture 1 holds, we can remove the no-circular-corridor constraint, which would prove the following slightly weaker conjecture:

Conjecture 2. Given any even cut graph, there exists a flat folding of the plane and a line on this folding such that scissor cutting along the line removes precisely the (folded) cut graph.

3.1 Consequences

This section describes two consequences of Conjecture 2.

What if the cut graph has some odd-degree vertices, but we are still only allowed to make scissor cuts? In [6], we prove that every planar bridgeless graph is the (nondisjoint) union of two even subgraphs. (A graph is *bridgeless* if it has no edges whose removal increases the number of connected components in the graph.) Assuming Conjecture 2, we can fold-and-scissor-cut the first subgraph; unfold, keeping the pieces together as before; and then fold-and-scissor-cut the second subgraph. In total, we make the entire graph with two scissor cuts. Hence, we have proved that Conjecture 2 implies the following:

Conjecture 3. Given any bridgeless cut graph, there exist two flat foldings of the plane and a line on each folding, such that scissor cutting along both lines in both foldings removes precisely the (folded) cut graph.

In his article on paper cutting, Martin Gardner [9] mentions an "unusual single-cut trick that is familiar to American magicians... known as the bicolor cut." The magician takes a thin piece of paper, colored red and black like an eight-by-eight checkerboard. After folding the paper flat, a single straight cut separates the red squares from the black squares, and simultaneously cuts out each square. Conjecture 2 implies a beautiful generalization of this magic trick.

Conjecture 4. Given any subdivision of a sheet of paper into red and black regions, there exists a flat folding of the paper and a line on the folding, such that all red regions are above the line, all black regions are below the line, and scissor cutting along the line separates all of the regions and no more.

Theorem 1 proves this conjecture for all red-black subdivisions that induce no circular corridors. Unfortunately, a checkerboard is not such a subdivision, so this result is not a generalization of the magic trick.

4 Crease Pattern

This section describes the basic crease pattern for our solution to the fold-and-cut problem, as well as the *default* mountain-valley assignment for some of these creases. The details of the folded state will be given in Section 5, where we will need, in the circular-corridor case, to add some more folds and reverse some creases.

The first collection of potential folds are the edges of the cut graph. More specifically, to satisfy the side assignment of faces above and below the cut line, we must fold along precisely those cut edges that are incident to faces assigned to the same side. By default, the fold is a valley between two faces above the cut line, and a mountain for two faces below the cut line.

The next section describes the *straight skeleton*, which is the main component for lining up the edges of the cut graph. To make the straight skeleton foldable, we add folds that are perpendicular to the cut edges in Section 4.2. Section 4.3 shows an interesting phenomenon in perpendiculars called spiraling. Finally, Section 4.4 studies the structures formed between perpendiculars, called corridors.

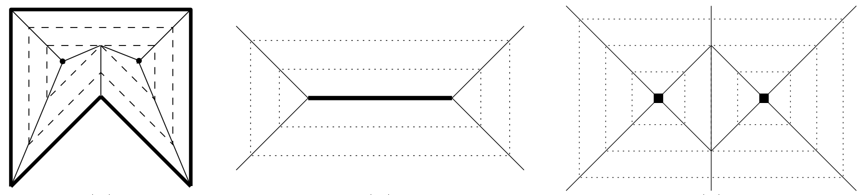

Fig. 6. *Examples of the straight skeleton: (a) Shrinking a single cut face. (b) A line segment. (c) Two points, with the squares chosen to be axis-parallel.*

4.1 Straight Skeleton

A natural way to line up two edges is to fold along the bisector of their extensions. A generalization of this idea to arbitrary cut graphs is the *straight skeleton.* This structure is defined to be the trajectories of the vertices as we *shrink* the faces of the cut graph. Note that "shrinking" the external face may seem more like "expanding." Formally, shrinking consists of continuously insetting each vertex towards the interior of the face, so that at any particular time, every shrunken edge is parallel to the original, and the perpendicular distance between the shrunken and original boundary edges is the same for all boundary edges. A face may split into multiple pieces, in which case we recursively shrink each piece. See Figure 6. A face may also become degenerate in the sense that two of its edges coincide to enclose a zero-area region; in this case, we include the edge as part of the straight skeleton.

Cut vertices of degrees zero and one must be treated specially; see Figure 6(b–c). A cut vertex of degree one is treated like an end of a rectangle with zero width. That is, we consider there to be an effective cut edge of length zero at the cut vertex, perpendicular to the incident cut edge. Similarly, a cut vertex of degree zero is treated like a square of zero area, with an arbitrarily chosen orientation. That is, we consider there to be four effective cut edges of length zero at the cut vertex, perpendicular in pairs to form a square.

The straight skeleton has only recently received thorough study. The basic idea goes back to at least 1984 [20, pp. 98–101]. The term "straight skeleton" was coined by Aichholzer et al. [2] in 1995, where it was only defined for the interior of a polygon. They are also the first to publish an algorithm for computing the straight skeleton, running in $O(n^2 \log n)$ time. The definition and this algorithm were extended to general plane graphs by Aichholzer and Aurenhammer [1] in 1996. Recently, Eppstein and Erickson have developed an $O(n^{17/11+\epsilon})$-time algorithm for general plane graphs [7].

The rest of this section describes some structure of the straight skeleton. Note that the straight skeleton is a plane graph. We will use "skeleton vertex," "skeleton edge," and "skeleton face" to refer to a vertex, edge, or face of the straight skeleton.

Globally define n to be the number of vertices in the cut graph.

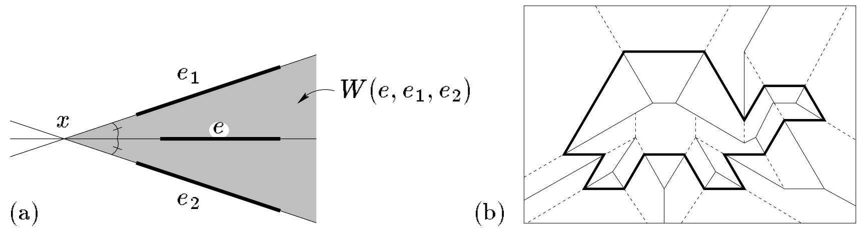

Fig. 7. *(a) The definition of an edge e bisecting two nonparallel edges e_1 and e_2. (b) Convex [reflex] portions of skeleton edges for the turtle are drawn solid [dashed].*

Lemma 1. [1] *The straight skeleton has $O(n)$ vertices, edges, and faces.*

Lemma 2. *Every cut edge is contained in exactly one skeleton face, and every skeleton face contains exactly one cut edge, if we include the zero-length cut edges formed by cut vertices of degrees zero and one.*

Proof. Straightforward. See [6]. □

Let $\overline{e}$ denote the line extending an edge e. Let e_1 and e_2 be two edges that do not intersect except possibly at a common endpoint. We say that an edge e *bisects* e_1 and e_2 if one of three cases holds. The first case is when $\overline{e}$, $\overline{e}_1$, and $\overline{e}_2$ are distinct and parallel, and $\overline{e}$ sits midway between the other two; that is, the perpendicular distance between $\overline{e}$ and $\overline{e}_1$ is positive and equals the perpendicular distance between $\overline{e}$ and $\overline{e}_2$. The second case is when $\overline{e}_1$ and $\overline{e}_2$ are the same line, and $\overline{e}$ is perpendicular to them.

Finally, the third case (see Figure 7(a)) is when $\overline{e}$, $\overline{e}_1$, and $\overline{e}_2$ are distinct and intersect at a common point x, and $\overline{e}$ bisects the angle of a certain wedge $W(e, e_1, e_2)$. This wedge is one of the four wedges between $\overline{e}_1$ and $\overline{e}_2$; furthermore, it is one of the two such wedges that contain a portion of $\overline{e}$. Specifically, $W(e, e_1, e_2)$ is defined to be the wedge between $\overline{e}_1$ and $\overline{e}_2$ that contains a portion of $\overline{e}$, and has all of e_1 or all of e_2 on its boundary. This definition of $W(e, e_1, e_2)$ is well defined by our assumption that e_1 and e_2 do not cross.

Lemma 3. *Let e be a skeleton edge, let f_1 and f_2 be the two incident skeleton faces, and let c_1 and c_2 be the cut edges contained in f_1 and f_2, respectively. Then e bisects c_1 and c_2.*

Proof. Straightforward. See [6]. □

As a step toward the mountain-valley assignment, we will now distinguish *convex* and *reflex* portions of skeleton edges; refer to Figure 7(b) for examples. Let e be a skeleton edge bisecting cut edges c_1 and c_2. We follow the cases in the definition of "bisect." If $\overline{c}_1$ and $\overline{c}_2$ are distinct and parallel, and $\overline{e}$ lies between them, then all of e is considered to be convex. If $\overline{c}_1$ and $\overline{c}_2$ are the same line, and $\overline{e}$ is perpendicular to them, then all of e is considered to be reflex. Finally, when $\overline{e}$ bisects $W(e, c_1, c_2)$, the portion of e inside the closed wedge $W(e, c_1, c_2)$ is considered to be convex, and the portion in the closed complement is considered to be reflex. Note that if the apex of $W(e, c_1, c_2)$ is in e, it is considered to be both convex and reflex.

The default mountain-valley assignment for a skeleton edge depends on the side assignment of its cut face. For "above" faces, convex portions are folded as mountains, and reflex portions are folded as valleys. For "below" faces, this assignment is reversed.

4.2 Perpendiculars

The straight skeleton by itself is clearly not foldable; in particular, its vertices typically have degree three, whereas a flat-foldable crease pattern only has vertices of even degree [4]. Intuitively, we can add a fold perpendicular to a cut edge, and maintain the property that the cut edges line up. What remains is to explicitly specify these folds, and how they interact with the straight skeleton.

The perpendicular associated with any point $p \in \mathbf{R}^2$ consists of a collection of line segments, rays, and lines called *perpendicular edges*, each associated with a skeleton face f. They are recursively defined as follows. For each closed skeleton face f that p is in, let l be the line going through p and perpendicular to (the line extending) the cut edge contained in face f. Let m be the connected piece of $l \cap f$ that touches p; this may be just p itself, a line segment of positive length, a ray, or a line. Then the perpendicular associated with p contains both m and the perpendiculars associated with the endpoints of m. We call m a *perpendicular edge associated with* f.

This completes the definition of perpendiculars. A *real perpendicular* is one that is incident to a skeleton vertex, and all other perpendiculars are called *imaginary*. We will only fold along real perpendiculars; imaginary perpendiculars will be useful for analyzing the structure of corridors in Section 5.1.

Examples of perpendiculars can be found throughout Figures 3 and 4.

In the definition of a perpendicular, we were careful to associate each perpendicular edge with a particular skeleton face. This is especially important for perpendicular edges that degenerate to points; we call these *zero-length* perpendicular edges. For example, incident to the middle skeleton vertex in Figure 6(a) are three perpendicular edges: an upward vertical ray, and two zero-length edges. Note that a zero-length perpendicular edge still has an orientation.

A natural question at this point is whether a perpendicular always consists of a finite number of line segments. The answer is no, and we discuss this situation in the next section. However, the number of real perpendiculars is finite:

Lemma 4. *There are $O(n)$ real perpendiculars.*

Proof. This follows by Lemma 1 because the number of real perpendiculars is at most the number of skeleton vertices. □

We now describe several properties of perpendiculars and perpendicular edges. A first observation is that the perpendicular edge associated with skeleton face

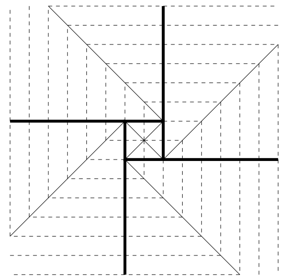

Fig. 8. *A simple example of spiraling.*

f is perpendicular to the cut edge contained in f; in particular, all such perpendicular edges are parallel.

Next let us demonstrate the ability to "walk around" with a pair of perpendicular edges in either of two directions.

Lemma 5. *Let e_1 and e_2 be two perpendicular edges associated with the same skeleton face. If e_1 and e_2 have ends v_1 and v_2, respectively, on a common skeleton edge s, then there are perpendicular edges e'_1 and e'_2, incident to v_1 and v_2 respectively, and associated with the other face incident to s, such that the perpendicular distance between e'_1 and e'_2 is the same as for e_1 and e_2.*

Proof. The existence of e'_1 and e'_2 follows from the definition of perpendiculars. The preservation of perpendicular distance follows because the skeleton edge s is a bisector of e_1 and e'_1, as well as e_2 and e'_2. □

Lemma 6. *Let p be a perpendicular edge in skeleton face f whose ends lie on the interior of skeleton edges e_1 and e_2. If p intersects the cut edge contained in f, then p hits e_1 and e_2 in their convex portions; otherwise, p hits one of e_1 and e_2 in its convex portion and the other in its reflex portion.*

Proof. See [6]. □

4.3 Spirals

One interesting phenomenon that can happen with perpendiculars is *spiraling.* A simple example is shown in Figure 8. The cut graph (drawn in thick lines) is an infinite "pinwheel." Each real perpendicular (drawn dashed) consists of an infinite number of edges in the whole plane, however:

Lemma 7. *Any bounded region of the plane is intersected by only finitely many real perpendicular edges.*

Proof (Sketch). Suppose to the contrary. We first argue that the perpendicular graph must contain an infinite path of degree-two vertices. In the plane, this path is a simple Jordan curve. Such an infinite Jordan curve can spiral or zig-zag, or

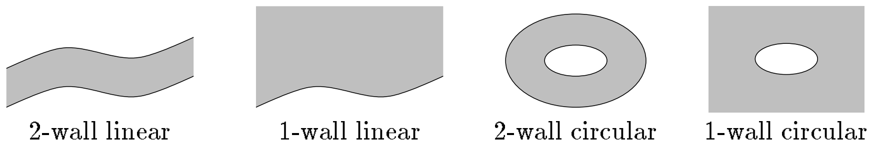

Fig. 9. The four possible shapes of corridors.

do any combination of these things. However, because perpendiculars can only bend due to the presence of skeleton vertices, and since there are only finitely many skeleton vertices, we are able to prove that our path must eventually spiral inward. By Lemma 5, consecutive rings of the spiral stay a constant width apart unless a skeleton vertex lies between the two rings, and again, this can happen only finitely often. Thus the inward spiral must eventually settle to some constant width, a contradiction. □

By Lemma 1, the straight skeleton consists of a finite number of creases, so the basic crease pattern is finite in any bounded convex region of the plane. Unfortunately, the number of creases is unbounded in terms of the number of vertices, minimum distance between two non-incident cut edges, or similar metric.

4.4 Corridors

Together, all of the real perpendicular edges form the *perpendicular graph*. This section describes the structure of *corridors*, which are simply the faces of the perpendicular graph. Unlike in usual plane graphs, a corridor never consists of a bounded simply connected region. See Figure 9 for the shapes that corridors may have. We characterize a corridor first by its topology: we call it *linear* if its interior is homeomorphic to an infinite band, and *circular* if its interior is homeomorphic to an annulus. Second, we characterize a corridor by the number of its walls, which can be one or two: a *wall* is one of the perpendiculars that bound the corridor. The wall count is equal to the number of connected components in $\mathbf{R}^2$ minus the interior of the corridor.

Let C be a corridor, f be a skeleton face, and C_f be the intersection of f with the interior of C. C_f may be disconnected, but each connected component of C_f has a *width* defined to be the minimum distance between two parallel lines that contain the region between them, and are perpendicular to the cut edge contained in f. By Lemma 5, corridors have *constant width* in the sense that the width as defined above is the same for all connected components of C_f, and for all choices of f. This is the motivation for the term "corridor." For one-wall corridors, the width is infinite; and for two-wall corridors, the width is the perpendicular distance between two parallel, minimally distant perpendicular edges bounding C.

The above claims are formalized in the following lemma.

Lemma 8. *Every corridor C is either linear or circular but not both, and has either one or two walls. Furthermore, C has constant width.*

Proof. Straightforward. See [6]. □

An example of a circular corridor is in the middle of the fancy star (Figure 4). Note that the inner wall consists just of zero-length perpendicular edges.

5 Folding

So far we have defined the basic crease pattern, consisting of skeleton edges, perpendicular edges, and some cut edges. We also have defined the default mountain-valley assignment for cut edges and skeleton edges. The goal of this section is to describe a flat folding of this crease pattern. Here we concentrate on the case where there are no circular corridors. In this case, the basic crease pattern is complete, and the default mountain-valley assignment is correct.

The problem splits naturally into two parts. In Section 5.1, we show how to fold a single corridor into an "accordion" using alternating mountain and valley folds at the skeleton edges crossing the corridor (see Figure 10). We show that this accordion folding lines up all the cut edges in the corridor. This is done locally, but we show that the foldings of two adjoining corridors are consistent.

In Section 5.2, we consider the global structure of how the corridors are joined. In case there are no circular corridors, this structure is a tree (see Figure 10). Each vertex of the tree corresponds to a perpendicular, and each edge of the tree corresponds to a corridor. If the paper lies originally in the xy plane, then the accordion folding maps each perpendicular to a line orthogonal to the xy plane, and maps each corridor to a strip of a plane orthogonal to the xy plane—the tree is precisely what we would see if we look at this folded structure from above, i.e. from the z direction. Because perpendiculars are orthogonal to cut edges, all the cut edges lie in the xy plane in this folded structure. The whole problem then reduces to the problem of folding a tree flat in the plane.

In the case of circular corridors, both of these steps need enhancing. First, a circular corridor cannot fold up into something as simple as an accordion; indeed, it may not be foldable at all if the side assignment is not constant. Second, the corridors are no longer joined in a tree-like fashion: it can be quite complex just to join two corridors together without crossings. The proof for the circular-corridor case is much more difficult, and deferred to [6].

5.1 Accordions

Consider a corridor, the creases intersecting it, and the mountain-valley assignment for those creases. We need to prove the existence of a folded state, which we call an *accordion* (see Figure 10), and we need to prove some properties of accordions. These results become obvious in the case where no crease terminates in the interior of a corridor. Skeleton edges pose no problem, but it is possible

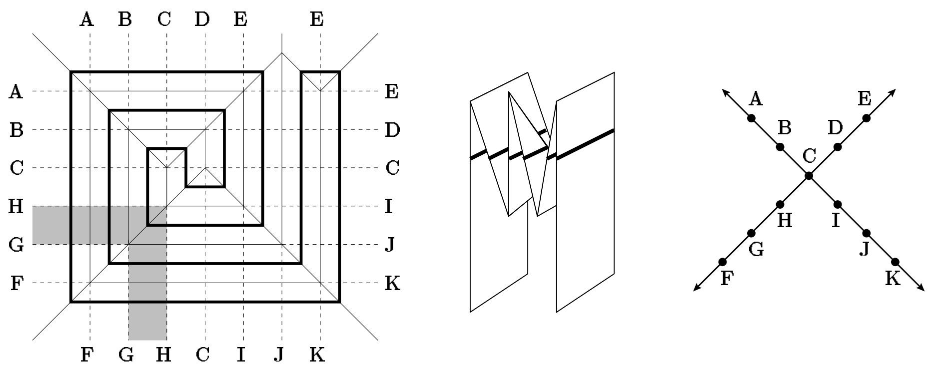

Fig. 10. *(Left) Full crease pattern for a spiral polygon. (Middle) The shaded corridor folded into an accordion. (Right) The tree model.*

for a cut edge to terminate in the interior of a corridor. For example, four cut edges terminate in the interior of the lower-left corridor in the turtle (Figure 4).

We can avoid this problem by adding the (imaginary) perpendiculars incident to cut vertices to our set of real perpendiculars, thus forming *subdivided corridors* with the property we want: folds cannot terminate interior to a subdivided corridor. Note that we do not fold along these added perpendiculars: they are being used only for our current purpose of describing the folded state of a corridor.

By Lemma 6, the creases in a subdivided corridor alternate between mountain and valley. Thus the accordion does not self-intersect.

A fold at a skeleton edge lines up the two cut edges bisected by the skeleton edge. (Note that the cut edges need not be in the corridor.) Because each wall of the (subdivided) corridor is perpendicular to the cut edges, folding at the skeleton edge causes the two incident perpendicular edges in the wall to fold to a common line. Of course this is also the case when we fold at a cut edge.

Hence, we have proved that when a corridor C is folded into an accordion, the cut edges intersecting C line up, and each wall of C lines up. The line(s) to which the wall(s) of C fold are called the *side(s)* of C; they are the places at which accordions join to each other.

We will orient accordions to be perpendicular to the xy plane.

Lemma 9. *Two adjacent corridors fold into accordions that match up at their common side.*

Proof. See [6]. □

5.2 Tree Model

We have shown that each accordion can be folded locally, and that they join compatibly, lining up all the cut edges. The problem thus reduces to folding at these joins so that the accordions lie on a common plane, resulting in the desired flat origami. This can be modeled by folding a *tree* in two dimensions; see Figure 10. Specifically, this tree corresponds to the xy projection of the folded model: each edge corresponds to an accordion, and each vertex corresponds to a side of an accordion. This model is certainly a tree, because a corridor either has one wall, in which case it corresponds to a leaf, or else removing the interior of a corridor from the plane leaves two disconnected pieces, in which case it corresponds to a bridge in the graph.

Finally, we need that every tree has a flat-folded state in the plane, that is, it can be folded to a line in the plane. This is straightforward; a proof can be found in [6]. Note that the mountain-valley assignment for perpendicular edges can be read from the tree folding. Our construction thus establishes

Theorem 2. *Given a cut graph that induces no circular corridors, and given any side assignment, there exists a flat folding of the plane that maps the cut edges and nothing else to a common line, and appropriately maps faces above or below this line.*

6 Conclusion

We have presented an algorithm that computes the crease pattern and the resulting flat origami that lines up a given plane graph. This allows one to fold a sheet of paper flat and make one complete straight cut to create any desired pattern of cuts. Concisely, folding and one straight cut suffice to make any plane graph. When only scissor cuts are allowed (that is, cuts cannot be made along folds) and the graph induces no circular corridors, we have shown that one scissor cut suffices for graphs whose vertices all have even degree, and two scissor cuts suffice for general graphs.

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References

1. O. Aichholzer and F. Aurenhammer. Straight skeletons for general polygonal figures in the plane. In *Proc. 2nd Int. Computing and Combinatorics Conf.*, pp. 117–126, Hong Kong, 1996.
2. O. Aichholzer, F. Aurenhammer, D. Alberts, and B. Gärtner. A novel type of skeleton for polygons. *JUCS*, 1(12):752–761, 1995.
3. M. Bern, E. Demaine, D. Eppstein, and B. Hayes. A disk-packing algorithm for an origami magic trick. In *Proc. Int. Conf. Fun with Algorithms*, Italy, June 1998.
4. M. Bern and B. Hayes. The complexity of flat origami. In *Proc. 7th ACM-SIAM Symp. Discrete Algorithms*, pp. 175–183, Atlanta, Jan. 1996.
5. E. D. Demaine and M. L. Demaine. Computing extreme origami bases. Tech. Rep. CS-97-22, University of Waterloo, May 1997.
6. E. D. Demaine, M. L. Demaine, and A. Lubiw. Folding and one straight cut suffice. Tech. Rep. CS-98-18, University of Waterloo, 1998.
7. D. Eppstein and J. Erickson. Raising roofs, crashing cycles, and playing pool: Applications of a data structure for finding pairwise interactions. In *Proc. 14th ACM Symp. Computational Geometry*, pp. 58–67, Minneapolis, June 1998.
8. G. N. Frederickson. Updates to *Dissections: Plane and Fancy*, July 1997. World Wide Web. `http://www.cs.purdue.edu/homes/gnf/book/Booknews/ch2.html`.
9. M. Gardner. Paper cutting. In *New Mathematical Diversions (Revised Edition)*, chapter 5, pp. 58–69. MAA, Washington, D.C., 1995.
10. R. Geretschläger. Euclidean constructions and the geometry of origami. *Math. Mag.*, 68(5):357–371, 1995.
11. H. Houdini. *Paper Magic*, pp. 176–177. E. P. Dutton & Company, 1922.
12. T. Hull. On the mathematics of flat origamis. *Congr. Numer.*, 100:215–224, 1994.
13. H. Huzita and B. Scimemi. The algebra of paper folding (origami). In *Proc. 1st Int. Meeting of Origami Science and Technology*, pp. 215–222, Ferrara, Dec. 1989.
14. J. Justin. Towards a mathematical theory of origami. In *Proc. 2nd Int. Meeting of Origami Science and Scientific Origami*, pp. 15–29, Otsu, Nov.–Dec. 1994.
15. T. Kawasaki. On the relation between mountain-creases and valley-creases of a flat origami. In *Proc. 1st Int. Meeting of Origami Science and Technology*, pp. 229–237, Ferrara, Dec. 1989.
16. D. König. *Theorie der endlichen und unendlichen Graphen.* Akademische Verlagsgesellschaft, Leipzig, 1936.
17. R. J. Lang. A computational algorithm for origami design. In *Proc. 12th Symp. Computational Geometry*, pp. 98–105, Philadelphia, May 1996.
18. R. J. Lang. *TreeMaker 4.0: A Program for Origami Design*, 1998.
19. G. M. Loe. *Paper Capers.* Magic, Inc., Chicago, 1955.
20. E. A. Lord and C. B. Wilson. *The Mathematical Description of Shape and Form.* Ellis Horwood Limited, West Sussex, England, 1984.
21. G. Nakamura. Personal communication, Dec. 1998.
22. K. C. Sen. *Wakoku Chiyekurabe (Mathematical Contests).* 1721.