The Stratified Foundations as a theory modulo

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Abstract

The *Stratified Foundations* are a restriction of naive set theory where the comprehension scheme is restricted to stratifiable propositions. It is known that this theory is consistent and that proofs strongly normalize in this theory.

Deduction modulo is a formulation of first-order logic with a general notion of cut. It is known that proofs normalize in a theory modulo if it has some kind of many-valued model called a *pre-model*.

We show in this note that the Stratified Foundations can be presented in deduction modulo and that the method used in the original normalization proof can be adapted to construct a pre-model for this theory.

The *Stratified Foundations* are a restriction of naive set theory where the comprehension scheme is restricted to stratifiable propositions. This theory is consistent [8] and proofs in this theory strongly normalize [2], while naive set theory is contradictory and the consistency of the Stratified Foundations together with the extensionality axiom - the so-called *New Foundations* - is open.

The Stratified Foundations extend simple type theory and the normalization proof for the Stratified Foundations, like that of type theory uses Girard's *reducibility candidates*. These two proofs, like all proofs following the line of Tait and Girard, have some parts in common. This motivates the investigation of general normalization theorems that have normalization theorems for specific theories as consequences. The normalization theorem for deduction modulo [7] is an example of such a general theorem. It concerns theories expressed in *deduction modulo* [5] that are first-order theories with a general notion of cut. According to this theorem, proofs normalize in a theory in deduction modulo if this theory has some kind of many-valued

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model called a *pre-model*. For instance, simple type theory can be expressed in deduction modulo [5, 6] and it has a pre-model [7, 6] and hence it has the normalization property. The normalization proof obtained this way is modular: all the lemmas specific to type theory are concentrated in the pre-model construction while the theorem that the existence of a pre-model implies normalization is generic and can be used for any other theory in deduction modulo.

The goal of this note is to show that the Stratified Foundations also can be presented in deduction modulo and that the method used in the original normalization proof can be adapted to construct a pre-model for this theory. The normalization proof obtained this way is simpler than the original one because it simply uses the fact that proofs normalize in the Stratified Foundations if this theory has a pre-model, while a variant of this proposition needs to be proved in the original proof.

It is worth noticing that the original normalization proof for the Stratified Foundations is already in two steps, where the first is the construction of a so-called *normalization model* and the second is a proof that proofs normalize in the Stratified Foundations if there is such a normalization model. Normalization models are more or less pre-models of the Stratified Foundations. So, we show that this notion of normalization model, that is specific to the Stratified Foundations, is an instance of a more general notion that can be defined for all theories modulo, and that the lemma that the existence of a normalization model implies normalization for the Stratified Foundations is an instance of a more general theorem that holds for all theories modulo.

The normalization proof obtained this way differs also from the original one in other respects. First, to remain in first-order logic, we do not use a presentation of the Stratified Foundations with a binder, but one with combinators. To express the Stratified Foundations with a binder in first-order logic, we could use de Bruijn indices and explicit substitutions along the lines of [6]. The pre-model construction below should generalize easily to such a presentation. Second, our cuts are cuts modulo, while the original proof uses Prawitz' *folding-unfolding* cut. It is shown in [4] that the normalization theorems are equivalent for the two notions of cuts, but that the notion of cut modulo is more general that the notion of folding-unfolding cut. Third, we use untyped reducibility candidates and not typed ones as in the original proof. This quite simplifies the technical details.

A last benefit of expressing the Stratified Foundations in deduction modulo is that we can use the method developed in [5] to organize proof search. The method obtained this way, that is an analog of higher-order resolution for the Stratified Foundations, is much more efficient than usual first-order proof search methods with the comprehension axiom, although it remains complete as the Stratified Foundations have the normalization property.

1 Deduction modulo

1.1 Identifying propositions

In deduction modulo, the notions of language, term and proposition are that of first-order logic. But, a theory is formed with a set of axioms Γ *and a congruence* $\equiv$ defined on propositions. Such a congruence may be defined by a rewrite systems on terms and on propositions (as propositions contain binders (quantifiers), these rewrite systems are in fact *combinatory reduction systems* [9]). Then, the deduction rules take this congruence into account. For instance, the *modus ponens* is not stated as usual

$$\frac{A \Rightarrow B \quad A}{B}$$

as the first premise need not be exactly $A \Rightarrow B$ but may be only congruent to this proposition, hence it is stated

$$\frac{C \quad A}{B} \text{ if } C \equiv A \Rightarrow B$$

All the rules of intuitionistic natural deduction may be stated in a similar way (figure 1). Classical deduction modulo is obtained by adding the excluded middle rule (figure 2).

For example, in arithmetic, we can define a congruence with the following rewrite system

$$0 + y \to y$$

$$S(x) + y \to S(x + y)$$

$$0 \times y \to 0$$

$$S(x) \times y \to x \times y + y$$

In the theory formed with a set of axioms Γ containing the axiom $\forall x\ x = x$ and this congruence, we can prove in natural deduction modulo, that the number 4 is even:

$$\cfrac{\cfrac{\cfrac{}{\Gamma \vdash_{\equiv} \forall x\ x = x}\text{ axiom}}{\Gamma \vdash_{\equiv} 2 \times 2 = 4}\ (x, x = x, 4)\ \forall\text{-elim}}{\Gamma \vdash_{\equiv} \exists x\ 2 \times x = 4}\ (x, 2 \times x = 4, 2)\ \exists\text{-intro}$$

$$\frac{}{\Gamma \vdash_{\equiv} B} \text{ axiom if } A \in \Gamma \text{ and } A \equiv B$$

$$\frac{\Gamma, A \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} C} \Rightarrow\text{-intro if } C \equiv (A \Rightarrow B)$$

$$\frac{\Gamma \vdash_{\equiv} C \quad \Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} B} \Rightarrow\text{-elim if } C \equiv (A \Rightarrow B)$$

$$\frac{\Gamma \vdash_{\equiv} A \quad \Gamma \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} C} \wedge\text{-intro if } C \equiv (A \wedge B)$$

$$\frac{\Gamma \vdash_{\equiv} C}{\Gamma \vdash_{\equiv} A} \wedge\text{-elim if } C \equiv (A \wedge B)$$

$$\frac{\Gamma \vdash_{\equiv} C}{\Gamma \vdash_{\equiv} B} \wedge\text{-elim if } C \equiv (A \wedge B)$$

$$\frac{\Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} C} \vee\text{-intro if } C \equiv (A \vee B)$$

$$\frac{\Gamma \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} C} \vee\text{-intro if } C \equiv (A \vee B)$$

$$\frac{\Gamma \vdash_{\equiv} D \quad \Gamma, A \vdash_{\equiv} C \quad \Gamma, B \vdash_{\equiv} C}{\Gamma \vdash_{\equiv} C} \vee\text{-elim if } D \equiv (A \vee B)$$

$$\frac{\Gamma \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} A} \perp\text{-elim if } B \equiv \perp$$

$$\frac{\Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} B} (x, A) \ \forall\text{-intro if } B \equiv (\forall x \ A) \text{ and } x \notin FV(\Gamma)$$

$$\frac{\Gamma \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} C} (x, A, t) \ \forall\text{-elim if } B \equiv (\forall x \ A) \text{ and } C \equiv [t/x]A$$

$$\frac{\Gamma \vdash_{\equiv} C}{\Gamma \vdash_{\equiv} B} (x, A, t) \ \exists\text{-intro if } B \equiv (\exists x \ A) \text{ and } C \equiv [t/x]A$$

$$\frac{\Gamma \vdash_{\equiv} C \quad \Gamma, A \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} B} (x, A) \ \exists\text{-elim if } C \equiv (\exists x \ A) \text{ and } x \notin FV(\Gamma B)$$

Figure 1: Natural deduction modulo

$$\frac{}{\Gamma \vdash_{\equiv} A} B \text{ Excluded middle if } A \equiv B \vee \neg B$$

Figure 2: Excluded middle

Substituting the variable x by the term 2 in the proposition $2 \times x = 4$ yields the proposition $2 \times 2 = 4$, that is congruent to $4 = 4$. The transformation of one proposition into the other, that requires several proof steps in usual natural deduction, is dropped from the proof in deduction modulo.

In this example, all the rewrite rules apply to terms. Deduction modulo permits also to consider rules rewriting atomic propositions to arbitrary ones. For instance, in the theory of integral domains, we have the rule

$$x \times y = 0 \to x = 0 \vee y = 0$$

that rewrites an atomic proposition to a disjunction.

Notice that, in the proof above, we do not need the axioms of addition and multiplication. Indeed, these axioms are now redundant: since the terms $0 + y$ and y are congruent, the axiom $\forall y\ 0 + y = y$ is congruent to the axiom of equality $\forall y\ y = y$. Hence, it can be dropped. Thus rewrite rules replace axioms.

This equivalence between rewrite rules and axioms is expressed in the the *equivalence lemma* that for every congruence $\equiv$, we can find a theory $\mathcal{T}$ such that $\Gamma \vdash_{\equiv} P$ is provable in deduction modulo if and only if $\mathcal{T}\Gamma \vdash P$ is provable in ordinary first-order logic [5]. Hence, deduction modulo is not a true extension of first-order logic, but rather an alternative formulation of first-order logic. Of course, the provable propositions are the same in both cases, but the proofs are very different.

1.2 Model of a theory modulo

A *model* of a congruence $\equiv$ is a model such that if $P \equiv Q$ then for all assignments, P and Q have the same denotation. A *model* of a theory modulo $\Gamma, \equiv$ is a model of the theory Γ and of the congruence $\equiv$. Unsurprisingly, the completeness theorem extends to classical deduction modulo [3] and a proposition P is provable in the theory $\Gamma, \equiv$ if and only if it is valid in all the models of $\Gamma, \equiv$.

1.3 Normalization in deduction modulo

Replacing axioms by rewrite rules in a theory changes the structure of proofs and in particular some theories may have the normalization property when expressed with axioms and not when expressed with rewrite rules. For instance, from the normalization theorem for first-order logic, we get that any proposition that is provable with the axiom $A \Leftrightarrow (B \wedge \neg A)$ has a normal proof. But if we transform this axiom into the rule $A \to B \wedge \neg A$ (Crabbé's rule [1]) the proposition $\neg B$ has a proof, but no normal proof.

We have proved a *normalization theorem*: proofs normalize in a theory modulo if this theory has a *pre-model* [7]. A pre-model is a many-valued model whose truth values are reducibility candidates, i.e. sets of proof-terms. Hence we first define proof-terms, then reducibility candidates and at last pre-models.

Definition 1.1 (Proof-term) Proof-terms *are inductively defined as follows.*

$$
\begin{array}{rl}
\pi ::= & \alpha \\
 & \mid \lambda\alpha\ \pi \mid (\pi\ \pi') \\
 & \mid \langle \pi, \pi' \rangle \mid fst(\pi) \mid snd(\pi) \\
 & \mid i(\pi) \mid j(\pi) \mid (\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3) \\
 & \mid (botelim\ \pi) \\
 & \mid \lambda x\ \pi \mid (\pi\ t) \\
 & \mid \langle t, \pi \rangle \mid (exelim\ \pi\ x\alpha\pi')
\end{array}
$$

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda\alpha\ \pi$ and $(\pi\ \pi')$ express proofs built with the introduction and elimination rules of the implication, terms of the form $\langle \pi, \pi' \rangle$ and $fst(\pi)$, $snd(\pi)$ express proofs built with the introduction and elimination rules of the conjunction, terms of the form $i(\pi), j(\pi)$ and $(\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3)$ express proofs built with the introduction and elimination rules of the disjunction, terms of the form $(botelim\ \pi)$ express proofs built with the elimination rule of the contradiction, terms of the form $\lambda x\ \pi$ and $(\pi\ t)$ express proofs built with the introduction and elimination rules of the universal quantifier and terms of the form $\langle t, \pi \rangle$ and $(exelim\ \pi\ x\alpha\pi')$ express proofs built with the introduction and elimination rules of the existential quantifier.

Definition 1.2 (Reduction) Reduction *on proof-terms is defined by the following rules that eliminate cuts step by step.*

$$(\lambda\alpha\ \pi_1\ \pi_2) \rhd [\pi_2/\alpha]\pi_1$$

$$fst(\langle \pi_1, \pi_2 \rangle) \rhd \pi_1$$

$$snd(\langle \pi_1, \pi_2 \rangle) \rhd \pi_2$$

$$(\delta\ i(\pi_1)\ \alpha\pi_2\ \beta\pi_3) \rhd [\pi_1/\alpha]\pi_2$$

$$(\delta\ j(\pi_1)\ \alpha\pi_2\ \beta\pi_3) \rhd [\pi_1/\beta]\pi_3$$

$$(\lambda x\ \pi\ t) \rhd [t/x]\pi$$

$$(exelim\ \langle t, \pi_1\rangle\ \alpha x\pi_2) \rhd [t/x, \pi_1/\alpha]\pi_2$$

Definition 1.3 (Reducibility candidates) *A proof-term is said to be* neutral *if it is a proof variable or an elimination (i.e. of the form* $(\pi\ \pi')$, $fst(\pi)$, $snd(\pi)$, $(\delta\ \pi_1\ \alpha\pi_2\ \beta\pi_3)$, $(botelim\ \pi)$, $(\pi\ t)$, $(exelim\ \pi\ x\alpha\pi')$*), but not an introduction. A set* R *of proof-terms is a* reducibility candidate *if*

- *if* $\pi \in R$*, then* π *is strongly normalizable,*
- *if* $\pi \in R$ *and* $\pi \rhd \pi'$ *then* $\pi' \in R$*,*
- *if* π *is neutral and if for every* π' *such that* $\pi \rhd^1 \pi'$, $\pi' \in R$ *then* $\pi \in R$.

We write $\mathcal{C}$ for the set of all reducibility candidates.

Definition 1.4 (Pre-model) *A* pre-model *for a language* $\mathcal{L}$ *is given by:*

- *a set* M*,*
- *for each function symbol* f *of arity* n *a function* $\hat{f}$ *from* M^n *to* M*,*
- *for each predicate symbol* Q *a function* $\hat{Q}$ *from* M^n *to* $\mathcal{C}$.

Definition 1.5 (Denotation in a pre-model) *Let* $\mathcal{P}$ *be a pre-model,* t *be a term and* φ *an assignment mapping all the free variables of* t *to elements of* M*. We define the object* $[\![t]\!]^{\mathcal{P}}_{\varphi}$ *by induction over the structure of* t.

- $[\![x]\!]^{\mathcal{P}}_{\varphi} = \varphi(x)$,
- $[\![f(t_1, \dots, t_n)]\!]^{\mathcal{P}}_{\varphi} = \hat{f}([\![t_1]\!]^{\mathcal{P}}_{\varphi}, \dots, [\![t_n]\!]^{\mathcal{P}}_{\varphi})$.

Let P *be a proposition and* φ *an assignment mapping all the free variables of* P *to elements of* M*. We define the reducibility candidate* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *by induction over the structure of* P.

- *If* P *is an atomic proposition* $Q(t_1, \dots, t_n)$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi} = \hat{Q}([\![t_1]\!]^{\mathcal{P}}_{\varphi}, \dots, [\![t_n]\!]^{\mathcal{P}}_{\varphi})$.
- *If* $P = Q \Rightarrow R$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of proofs* π *such that* π *is strongly normalizable and whenever it reduces to* $\lambda\alpha\pi_1$ *then for every* π' *in* $[\![Q]\!]^{\mathcal{P}}_{\varphi}$, $[\pi'/\alpha]\pi_1$ *is in* $[\![R]\!]^{\mathcal{P}}_{\varphi}$.

- *If* $P = Q \wedge R$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of proofs* π *such that* π *is strongly normalizable and whenever it reduces to* $\langle \pi_1, \pi_2 \rangle$ *then* π_1 *is in* $[\![Q]\!]^{\mathcal{P}}_{\varphi}$ *and* π_2 *is in* $[\![R]\!]^{\mathcal{P}}_{\varphi}$.

- *If* $P = Q \vee R$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of proofs* π *such that* π *is strongly normalizable and whenever it reduces to* $i(\pi_1)$ *(resp.* $j(\pi_2)$*) then* π_1 *(resp.* π_2*) is in* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *(resp.* $[\![Q]\!]^{\mathcal{P}}_{\varphi}$*).*

- *If* $P = \bot$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of strongly normalizable proofs.*

- *If* $P = \forall x\ Q$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of proofs* π *such that* π *is strongly normalizable and whenever it reduces to* $\lambda x\ \pi_1$ *then for every term* t *and every element* a *of* M $[t/x]\pi_1$ *is in* $[\![Q]\!]^{\mathcal{P}}_{\varphi+a/x}$.

- *If* $P = \exists x\ Q$ *then* $[\![P]\!]^{\mathcal{P}}_{\varphi}$ *is the set of proofs* π *such that* π *is strongly normalizable and whenever it reduces to* $\langle t, \pi_1 \rangle$ *then there exists an element* a *in* M *such that* π_1 *is in* $[\![Q]\!]^{\mathcal{P}}_{\varphi+a/x}$.

Definition 1.6 *A pre-model is said to be a* pre-model of a congruence $\equiv$ *if when* $A \equiv B$ *then for every assignment* φ, $[\![A]\!]^{\mathcal{P}}_{\varphi} = [\![B]\!]^{\mathcal{P}}_{\varphi}$.

Theorem 1.1 (Normalization) *[7] If a congruence* $\equiv$ *has a pre-model all proofs modulo* $\equiv$ *strongly normalize.*

2 The Stratified Foundations

2.1 The Stratified Foundations as a first-order theory

Definition 2.1 *(Stratifiable proposition)*

A proposition P *in the language* $\in$ *is said to be* stratifiable *if there exists a function* S *mapping every variable (bound or free) of* P *to a natural number in such a way that every atomic proposition of* P, $x \in y$ *is such that* $S(y) = S(x) + 1$.

For instance, the proposition

$$\forall v\ (v \in x \Leftrightarrow v \in y) \Rightarrow \forall w\ (x \in w \Rightarrow y \in w)$$

is stratifiable (take, for instance, $S(v) = 4$, $S(x) = S(y) = 5$, $S(w) = 6$) but not the proposition

$$\forall v\ (v \in x \Leftrightarrow v \in y) \Rightarrow x \in y$$

Definition 2.2 *(The stratified comprehension scheme)*

For every stratifiable proposition P whose free variables are among $x_1, \ldots, x_n, x_{n+1}$ we take the axiom

$$\forall x_1 \ \ldots \ \forall x_n \ \exists z \ \forall x_{n+1} \ (x_{n+1} \in z \Leftrightarrow P)$$

Definition 2.3 *(The skolemized stratified comprehension scheme)*

When we skolemize this axiom we introduce for each stratifiable proposition P in the language $\in$ and sequence of variables $x_1, \ldots, x_n, x_{n+1}$ such that the free variables of P are among $x_1, \ldots, x_n, x_{n+1}$, a function symbol $f_{x_1,\ldots,x_n,x_{n+1},P}$ and the axiom

$$\forall x_1 \ \ldots \ \forall x_n \ \forall x_{n+1} \ (x_{n+1} \in f_{x_1,\ldots,x_n,x_{n+1},P}(x_1, \ldots, x_n) \Leftrightarrow P)$$

2.2 The Stratified Foundations as a theory modulo

Now we want to replace the axiom scheme above by a rewrite rule, defining a congruence on propositions, so that the Stratified Foundations are defined as an axiom free theory modulo.

Definition 2.4 *(The rewrite system $\mathcal{R}$)*

$$t_{n+1} \in f_{x_1,\ldots,x_n,x_{n+1},P}(t_1, \ldots, t_n) \to [t_1/x_1, \ldots, t_n/x_n, t_{n+1}/x_{n+1}]P$$

Proposition 2.1 *The rewrite system $\mathcal{R}$ is confluent.*

Proof. It is an orthogonal combinatory reduction system [9]. □

Proposition 2.2 *The rewrite system $\mathcal{R}$ is terminating.*

Proof. If A is an atomic proposition we write $\|A\|$ for the number of function symbols in A. If A is a proposition, containing the atomic propositions $A_1, \ldots, A_p$ we write A° for the multiset of natural numbers $\{\|A_1\|, \ldots, \|A_p\|\}$. We show that if a proposition A reduces in one step to a proposition B then $B^\circ < A^\circ$ for the multiset ordering.

If the proposition A reduces in one step to B, there is an atomic proposition of A, say A_1, that has the form $t_{n+1} \in f_{x_1,\ldots,x_n,x_{n+1},P}(t_1, \ldots, t_n)$ and reduces to $B_1 = [t_1/x_1, \ldots, t_n/x_n, t_{n+1}/x_{n+1}]P$. Every atomic proposition b of B_1 has the form $[t_1/x_1, \ldots, t_n/x_n, t_{n+1}/x_{n+1}]a$ where a is an atomic proposition of P. The proposition a has the form $x_i \in x_j$ for distinct i and j (since P is stratifiable). Hence b has the form $t_i \in t_j$ and $\|b\| < \|A_1\|$. Therefore $B^\circ < A^\circ$. □

Proposition 2.3 *A proposition P is provable from the skolemized comprehension scheme if and only if it is provable modulo the rewrite system $\mathcal{R}$.*

2.3 Consistency

2.3.1 Automorphisms of models of set theory

If $\mathcal{M}$ is a model of set theory we write M for the set of elements of the model, $\in_{\mathcal{M}}$ for the denotation of the symbol $\in$ in this model, $\wp_{\mathcal{M}}$ for the powerset in this model, etc. We write also $[\![P]\!]^{\mathcal{M}}_{\varphi}$ for the denotation of a proposition P for the assignment φ.

The proof of the consistency of the Stratified Foundations rests on the existence of a model of Zermelo's set theory, such that there is a bijection σ from M to M and a family v_i of elements of M, $i \in \mathbb{Z}$ such that

$$a \in_{\mathcal{M}} b \text{ if and only if } \sigma a \in_{\mathcal{M}} \sigma b$$

$$\sigma v_i = v_{i+1}$$

$$v_i \subseteq_{\mathcal{M}} v_{i+1}$$

$$\wp_{\mathcal{M}}(v_i) \subseteq_{\mathcal{M}} v_{i+1}$$

The existence of such a model is proved in [8].

Using the fact that $\mathcal{M}$ is a model of the axiom of extensionality, we prove that $a \subseteq_{\mathcal{M}} b$ if and only if $\sigma a \subseteq_{\mathcal{M}} \sigma b$, $\sigma\{a, b\}_{\mathcal{M}} = \{\sigma a, \sigma b\}_{\mathcal{M}}$, $\sigma\langle a, b\rangle_{\mathcal{M}} = \langle \sigma a, \sigma b\rangle_{\mathcal{M}}$, $\sigma\wp(a) = \wp(\sigma a)$, etc.

For the normalization proof, we will further need that $\mathcal{M}$ is an ω-model. We define $\overline{0} = \emptyset_{\mathcal{M}}$, $\overline{n+1} = \overline{n} \cup_{\mathcal{M}} \{\overline{n}\}_{\mathcal{M}}$. A ω-model is a model such that $a \in_{\mathcal{M}} \mathbb{N}_{\mathcal{M}}$ if and only if there exists n in $\mathbb{N}$ such that $a = \overline{n}$. The existence of such a model is also proved in [8] (see also [2]).

Using the fact that $\mathcal{M}$ is a model of the axiom of extensionality, we prove that $\sigma\emptyset_{\mathcal{M}} = \emptyset_{\mathcal{M}}$ and then, by induction on n that $\sigma\overline{n} = \overline{n}$.

Notice that since $\wp_{\mathcal{M}}(v_i) \subseteq_{\mathcal{M}} v_{i+1}$, $\emptyset_{\mathcal{M}} \in_{\mathcal{M}} v_i$ and for all n, $\overline{n} \in_{\mathcal{M}} v_i$. Hence as the model is an ω-model $\mathbb{N}_{\mathcal{M}} \subseteq_{\mathcal{M}} v_i$.

In an ω-model, we can identify the set $\mathbb{N}$ of natural numbers with the set of objects a in $\mathcal{M}$ such that $a \in_{\mathcal{M}} \mathbb{N}_{\mathcal{M}}$. To each proof-term we can associate a natural number n (its Gödel number) and then the element $\overline{n}$ of $\mathcal{M}$. Proof-terms, their Gödel number and the encoding of this number in $\mathcal{M}$ will be identified in the following.

2.3.2 A model

Let U be the set of elements a of M such that $a \in_{\mathcal{M}} v_0$.

Definition 2.5 *Let P be a proposition in the language $\in$ and φ be an assignment. We define the value $[P]_{\varphi}$ as follows.*

- *If* $P = x_i \in x_j$ *then* $[P]_\varphi = 1$ *if* $\varphi(x_i) \in_{\mathcal{M}} \sigma\varphi(x_j)$ *and* $[P]_\varphi = 0$ *otherwise.*

- *If* $P = Q \Rightarrow R$ *then* $[P]_\varphi = 1$ *if* $[Q]_\varphi = 0$ *or* $[R]_\varphi = 1$ *and* $[P]_\varphi = 0$ *otherwise.*

- *If* $P = Q \wedge R$ *then* $[P]_\varphi = 1$ *if* $[Q]_\varphi = 1$ *and* $[R]_\varphi = 1$ *and* $[P]_\varphi = 0$ *otherwise.*

- *If* $P = Q \vee R$ *then* $[P]_\varphi = 1$ *if* $[Q]_\varphi = 1$ *or* $[R]_\varphi = 1$ *and* $[P]_\varphi = 0$ *otherwise.*

- *if* $P = \bot$ *then* $[P]_\varphi = 0$.

- *If* $P = \forall x\ Q$ *then* $[P]_\varphi = 1$ *if for all* a *in* U, $[Q]_{\varphi+a/x} = 1$ *and* $[P]_\varphi = 0$ *otherwise.*

- *If* $P = \exists x\ Q$ *then* $[P]_\varphi = 1$ *if there exists* a *in* U *such that* $[Q]_{\varphi+a/x} = 1$ *and* $[P]_\varphi = 0$ *otherwise.*

Proposition 2.4 *For every stratifiable proposition* P *whose free variables are among* $x_1, \ldots, x_n, x_{n+1}$ *and for all* $a_1, \ldots, a_n$ *in* U, *there exists an element* b *in* U *such that for every* a_{n+1} *in* U, $a_{n+1} \in_{\mathcal{M}} \sigma b$ *if and only if* $[P]_{a_1/x_1,\ldots,a_n/x_n,a_{n+1}/x_{n+1}} = 1$

Proof. Let $|P|$ be the proposition defined as follows.

- $|P| = P$ if P is atomic,

- $|P \Rightarrow Q| = |P| \Rightarrow |Q|$, $|P \wedge Q| = |P| \wedge |Q|$, $|P \vee Q| = |P| \vee |Q|$, $|\bot| = \bot$,

- $|\forall x\ P| = \forall x\ ((x \in E_{S(x)}) \Rightarrow |P|)$,

- $|\exists x\ P| = \exists x\ ((x \in E_{S(x)}) \wedge |P|)$.

Notice that the free variables of $|P|$ are among $E_0, \ldots, E_m, x_1, \ldots, x_n, x_{n+1}$. We let

$$\varphi = a_1/x_1, \ldots, a_n/x_n, a_{n+1}/x_{n+1}$$

$$\psi = v_0/E_0, \ldots, v_m/E_m, \sigma^{k_1}a_1/x_1, \ldots, \sigma^{k_n}a_n/x_n, \sigma^{k_{n+1}}a_{n+1}/x_{n+1}$$

where $k_1 = S(x_1), \ldots, k_{n+1} = S(x_{n+1})$. We check, by induction over the structure of P, that if P is a stratifiable proposition, then

$$[\![|P|]\!]^{\mathcal{M}}_{\psi} = [P]_\varphi$$

- If P is an atomic proposition $x_i \in x_j$, then $k_j = k_i + 1$, $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $\sigma^{k_i} a_i \in_{\mathcal{M}} \sigma^{k_j} a_j$ if and only if $a_i \in_{\mathcal{M}} \sigma a_j$, if and only if $[P]_\varphi = 1$.

- if $P = Q \Rightarrow R$ then $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[\![|Q|]\!]_\psi^{\mathcal{M}} = 0$ or $[\![|R|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[Q]_\varphi = 0$ or $[R]_\varphi = 1$ if and only if $[P]_\varphi = 1$.

- if $P = Q \wedge R$ then $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[\![|Q|]\!]_\psi^{\mathcal{M}} = 1$ and $[\![|R|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[Q]_\varphi = 1$ and $[R]_\varphi = 1$ if and only if $[P]_\varphi = 1$.

- if $P = Q \vee R$ then $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[\![|Q|]\!]_\psi^{\mathcal{M}} = 1$ or $[\![|R|]\!]_\psi^{\mathcal{M}} = 1$ if and only if $[Q]_\varphi = 1$ and $[R]_\varphi = 1$ if and only if $[P]_\varphi = 1$.

- $[\![|\bot|]\!]_\psi^{\mathcal{M}} = 0 = [\bot]_\varphi$.

- if $P = \forall x\ Q$ then $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if for every c in $\mathcal{M}$ such that $c \in_{\mathcal{M}} v_k$, $[\![|Q|]\!]_{\psi+c/x}^{\mathcal{M}} = 1$, if and only if for every e in U, $[\![|Q|]\!]_{\psi+\sigma^k e/x}^{\mathcal{M}} = 1$ if and only if for all e in U, $[Q]_{\varphi+e/x} = 1$ if and only if $[P]_\varphi = 1$.

- if $P = \exists x\ Q$ then $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ if and only if there exists c in $\mathcal{M}$ such that $c \in_{\mathcal{M}} v_k$ and $[\![|Q|]\!]_{\psi+c/x}^{\mathcal{M}} = 1$, if and only if there exists e in U such that $[\![|Q|]\!]_{\psi+\sigma^k e/x}^{\mathcal{M}} = 1$ if and only if there exists e in U such that $[Q]_{\varphi+e/x} = 1$ if and only if $[P]_\varphi = 1$.

Then, the model $\mathcal{M}$ is a model of the comprehension scheme. Hence, it is a model of the proposition

$$\forall E_0\ \ldots\ \forall E_m\ \forall x_1\ \ldots\ \forall x_n\ \forall y\ \exists z\ \forall x_{n+1}\ (x_{n+1} \in z \Leftrightarrow (x_{n+1} \in y \wedge |P|))$$

i.e.

$$[\![\forall E_0\ \ldots\ \forall E_m\ \forall x_1\ \ldots\ \forall x_n\ \forall y\ \exists z\ \forall x_{n+1}\ (x_{n+1} \in z \Leftrightarrow (x_{n+1} \in y \wedge |P|))]\!]^{\mathcal{M}} = 1$$

Hence, there exists an object b_0 such that

$$[\![(x_{n+1} \in z \Leftrightarrow (x_{n+1} \in y \wedge |P|))]\!]_{\psi+v_{k_{n+1}}/y+b_0/z}^{\mathcal{M}} = 1$$

We have $\sigma^{k_{n+1}} a_{n+1} \in_{\mathcal{M}} b_0$ if and only if $\sigma^{k_{n+1}} a_{n+1} \in_{\mathcal{M}} v_{k_{n+1}}$ and $[\![|P|]\!]_\psi^{\mathcal{M}} = 1$ thus $a_{n+1} \in_{\mathcal{M}} \sigma^{-k_{n+1}} b_0$ if and only if a_{n+1} is in U and $[P]_\varphi = 1$. We take $b = \sigma^{-(k_{n+1}+1)} b_0$. For all a_{n+1} in U, we have $a_{n+1} \in_{\mathcal{M}} \sigma b$ if and only if $[P]_\varphi = 1$.

Notice finally that $b_0 \in_{\mathcal{M}} \wp_{\mathcal{M}}(v_{k_{n+1}})$, thus $b_0 \in_{\mathcal{M}} v_{k_{n+1}+1}$, $b \in_{\mathcal{M}} v_0$ and hence b is in U. $\Box$

Definition 2.6 (Jensen's model) *The model $\mathcal{U} = \langle U, \in_{\mathcal{U}}, \hat{f}_{x_1,\ldots,x_n,y,P}\rangle$ is defined as follows. The base set is U. The relation $\in_{\mathcal{U}}$ is defined by $a \in_{\mathcal{U}} b$ if and only if $a \in_{\mathcal{M}} \sigma b$. The function $\hat{f}_{x_1,\ldots,x_n,x_{n+1},P}$ maps $(a_1,\ldots,a_n)$ to an object b such that for all a_{n+1} in U, $a_{n+1} \in_{\mathcal{M}} \sigma b$ if and only if $[P]_{a_1/x_1,\ldots,a_n/x_n,a_{n+1}/x_{n+1}}$.*

Proposition 2.5 *The model $\mathcal{U}$ is a model of the Stratified Foundations.*

Proof. By induction over the structure of P, if P is a proposition in the language $\in$, its denotation in $\mathcal{U}$ for the assignment φ is $[P]_\varphi$. Then

$$[\![t_{n+1} \in f_{x_1,\ldots,x_n,x_{n+1},P}(t_1,\ldots,t_n)]\!]^{\mathcal{U}}_\varphi = 1$$

if and only if

$$[\![t_{n+1}]\!]^{\mathcal{U}}_\varphi \in_{\mathcal{M}} \sigma \hat{f}_{x_1,\ldots,x_n,x_{n+1},P}([\![t_1]\!]^{\mathcal{U}}_\varphi,\ldots,[\![t_n]\!]^{\mathcal{U}}_\varphi)$$

if and only if

$$[P]_{[\![t_1]\!]^{\mathcal{U}}_\varphi/x_1,\ldots,[\![t_n]\!]^{\mathcal{U}}_\varphi/x_n,[\![t_{n+1}]\!]^{\mathcal{U}}_\varphi/x_{n+1}} = 1$$

if and only if

$$[\![P]\!]^{\mathcal{U}}_{[\![t_1]\!]^{\mathcal{U}}_\varphi/x_1,\ldots,[\![t_n]\!]^{\mathcal{U}}_\varphi/x_n,[\![t_{n+1}]\!]^{\mathcal{U}}_\varphi/x_{n+1}} = 1$$

if and only if

$$[\![[t_1/x_1,\ldots,t_n/x_n,t_{n+1}/x_{n+1}]P]\!]^{\mathcal{U}}_\varphi = 1$$

Hence, if $A \equiv B$ then A and B have the same denotation. □

2.4 Normalization

We now want to construct a pre-model for the Stratified Foundations.

Let $u_i = v_{3i}$ and $\tau = \sigma^3$. The function τ is an automorphism of $\mathcal{M}$, $\tau u_i = u_{i+1}$, $u_i \subseteq_{\mathcal{M}} u_{i+1}$ and $\wp_{\mathcal{M}}\ (\wp_{\mathcal{M}}\ (\wp_{\mathcal{M}}\ (u_i))) \subseteq_{\mathcal{M}} u_{i+1}$.

Notice that as $\mathcal{M}$ is an ω-model, for each recursively enumerable relation R on natural numbers, there is an object r in $\mathcal{M}$ such that $R(a_1,\ldots,a_n)$ if and only if $\langle a_1,\ldots,a_n\rangle_{\mathcal{M}} \in_{\mathcal{M}} r$. In particular there is

- an object $Proof$ such that $\pi \in_{\mathcal{M}} Proof$ if and only if π is (the encoding in $\mathcal{M}$ of the Gödel number of) a proof,
- an object $Term$ such that $t \in_{\mathcal{M}} Term$ if and only if t is (the encoding of the Gödel number of) a term,

- an object $Subst$ such that $\langle \pi, \alpha, \pi_1, \pi_2 \rangle_{\mathcal{M}} \in_{\mathcal{M}} Subst$ if and only if π, π_1 and π_2 are (encodings of Gödel numbers of) proofs, α is (the encoding of the Gödel number of) a proof variable and $\pi = [\pi_2/\alpha]\pi_1$,
- an object $Subst'$ such that $\langle \pi, x, \pi_1, t \rangle_{\mathcal{M}} \in_{\mathcal{M}} Subst'$ if and only if π and π_1 are (encodings of the Gödel numbers of) proofs, x is (the encoding of the Gödel number of) a term variable and t (the encoding of the Gödel number of) a term and $\pi = [t/x]\pi_1$,
- an object Red such that $\langle \pi, \pi_1 \rangle_{\mathcal{M}} \in_{\mathcal{M}} Red$ if and only if π and π' are (encodings of Gödel numbers of) proofs and $\pi \rhd^* \pi_1$,
- an object Sn such that $\pi \in_{\mathcal{M}} Sn$ if and only if π is (the encoding of the Gödel number of) a strongly normalizable proof,
- an object $AndI$ such that $\langle \pi, \pi_1, \pi_2 \rangle_{\mathcal{M}} \in_{\mathcal{M}} AndI$ if and only if π, π_1 and π_2 are (encodings of Gödel numbers of) proofs and $\pi = \langle \pi_1, \pi_2 \rangle$,
- an object $OrI1$ (resp. $OrI2$) such that $\langle \pi, \pi_1 \rangle_{\mathcal{M}} \in_{\mathcal{M}} OrI1$ (resp. $\langle \pi, \pi_2 \rangle_{\mathcal{M}} \in_{\mathcal{M}} OrI2$) if and only if π and π_1 (resp. π and π_2) are (encodings of Gödel numbers of) proofs and $\pi = i(\pi_1)$ (resp. $\pi = j(\pi_2)$),
- an object $ForallI$ such that $\langle \pi, \alpha, \pi_1 \rangle_{\mathcal{M}} \in_{\mathcal{M}} ForallI$ if and only π and π_1 are (encodings of Gödel numbers of) proofs, α is (the encoding of the Gödel number of) a proof variable, and $\pi = \lambda\alpha\pi_1$,
- an object $ExistsI$ such that $\langle \pi, t, \pi_1 \rangle_{\mathcal{M}} \in_{\mathcal{M}} ExistsI$ if and only if π and π_1 are (encodings of Gödel numbers of) proofs, t is (the encoding of the Gödel number of) a term and $\pi = \langle t, \pi_1 \rangle$.

Notice also that, since $\mathcal{M}$ is a model of the comprehension scheme, there is an object Cr such that $\alpha \in_{\mathcal{M}} Cr$ if and only if α is a reducibility candidate (i.e. the set of objects β such that $\beta \in_{\mathcal{M}} \alpha$ is a reducibility candidate).

Definition 2.7 (Admissible) *An element α of M is said to* admissible *at level i if α is a set of pairs $\langle \pi, \beta \rangle_{\mathcal{M}}$ where π is a proof and β an element of u_i and and for each β in u_i the set of π such that $\langle \pi, \beta \rangle_{\mathcal{M}} \in_{\mathcal{M}} \alpha$ is a reducibility candidate.*

Notice that if R is a reducibility candidate (for instance the set of all strongly normalizable proofs) then the set $R \times_{\mathcal{M}} u_i$ is admissible at level i. Hence, for each integer i, there are elements of $\mathcal{M}$ admissible at level i.

Proposition 2.6 *There is an element A_i in M such that $\alpha \in_{\mathcal{M}} A_i$ if and only if α is admissible at level i.*

Proof. An element α of $\mathcal{M}$ admissible at level i if and only if

$$\alpha \in_{\mathcal{M}} \wp_{\mathcal{M}}(Proof \times_{\mathcal{M}} u_i)$$
$$\wedge \forall \beta \ (\beta \in_{\mathcal{M}} u_i \Rightarrow \exists C \ (C \in_{\mathcal{M}} Cr \wedge (\langle \pi, \beta \rangle_{\mathcal{M}} \in_{\mathcal{M}} \alpha \Leftrightarrow \pi \in_{\mathcal{M}} C)))$$

Hence, as $\mathcal{M}$ is a model of the comprehension scheme, there is an element A_i in M such that $\alpha \in_{\mathcal{M}} A_i$ if and only if α is admissible at level i. $\Box$

Notice that $\alpha \in \tau A_i$ if and only if $\alpha \in A_{i+1}$. Hence as $\mathcal{M}$ is a model of the extensionality axiom, $\tau A_i = A_{i+1}$.

Notice, at last, that $A_i \subseteq_{\mathcal{M}} \wp_{\mathcal{M}}(Proof \times_{\mathcal{M}} u_i) \subseteq_{\mathcal{M}} \wp_{\mathcal{M}}(u_i \times_{\mathcal{M}} u_i) \subseteq_{\mathcal{M}} \wp_{\mathcal{M}} \ (\wp_{\mathcal{M}} \ (\wp_{\mathcal{M}} \ (u_i))) \subseteq_{\mathcal{M}} u_{i+1}$.

Proposition 2.7 *If $\beta \in_{\mathcal{M}} A_i$ and $\alpha \in_{\mathcal{M}} A_{i+1}$ then the set of π such that $\langle \pi, \beta \rangle \in_{\mathcal{M}} \alpha$ is a reducibility candidate.*

Proof. As $\alpha \in_{\mathcal{M}} A_{i+1}$ and $\beta \in_{\mathcal{M}} A_i \subseteq_{\mathcal{M}} u_{i+1}$, the set of π such that $\langle \pi, \beta \rangle \in_{\mathcal{M}} \alpha$ is a reducibility candidate. $\Box$

Let N be the set of elements of M that are admissible at level 0.

Definition 2.8 *Let P be a proposition in the language $\in$, φ be an assignment, we define the set $[P]_\varphi$ of elements of $\mathcal{M}$ as follows.*

- *If $P = x_i \in x_j$ then $[P]_\varphi$ is the set of proofs π such that $\langle \pi, \varphi(x_i) \rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau\varphi(x_j)$.*
- *If $P = Q \Rightarrow R$ then $[P]_\varphi$ is the set of proofs π such that π is strongly normalizable and π reduces to $\lambda\alpha\pi_1$ then for every π' in $[A]_\varphi$, $[\pi'/\alpha]\pi_1$ is in $[B]_\varphi$.*
- *If $P = Q \wedge R$ then $[P]_\varphi$ is the set of π such that π is strongly normalizable and whenever π reduces to $\langle \pi_1, \pi_2 \rangle$ then π_1 is in $[Q]_\varphi$ and π_2 is in $[R]_\varphi$.*
- *If $P = Q \vee R$ then $[P]_\varphi$ is the set of π such that π is strongly normalizable and whenever π reduces to $i(\pi_1)$ (resp. $j(\pi_2)$) then π_1 (resp. π_2) is in $[A]_\varphi$ (resp. $[B]_\varphi$).*
- *If $P = \perp$ then $[P]_\varphi$ is the set of strongly normalizable proofs.*
- *If $P = \forall x \ Q$ then $[P]_\varphi$ is the set of proofs π such that π is strongly normalizable and whenever π reduces to $\lambda x \ \pi'$ then for every element a in N and every term t, $\pi'[t/x]$ is in $[P]_{\varphi+a/x}$.*

- *If $P = \exists x\ Q$ then $[P]_\varphi$ is the set of proofs π such that π is strongly normalizable and whenever π reduces to $\langle t, \pi_1 \rangle$ then there exists an element a in N such that π_1 is in $[A]_{\varphi + a/x}$.*

Proposition 2.8 *For every stratifiable proposition P whose free variables are among $x_1, \ldots, x_n, x_{n+1}$ and for all $a_1, \ldots, a_n$ in N, there exists an element b in N such that for every a_{n+1} in N, $\langle \pi, a_{n+1} \rangle_\mathcal{M} \in_\mathcal{M} \tau b$ if and only if π is in $[P]_{a_1/x_1, \ldots, a_{n+1}/x_{n+1}}$.*

Proof. Let $|P|$ be the proposition (read p *realizes* P) defined as follows.

- $|x_i \in x_j| = \langle p, x_i \rangle \in x_j$,
- $|P \Rightarrow Q| = p \in sn \wedge \forall q\ \forall w\ \forall r\ (\langle p, q \rangle \in red \wedge \langle q, w, r \rangle \in impI) \Rightarrow \forall s\ [s/p]|P| \Rightarrow \forall t\ \langle t, r, w, s \rangle \in subst \Rightarrow [t/p]|Q|)$,
- $|P \wedge Q| = p \in sn \wedge \forall q\ \forall r\ \forall s\ ((\langle p, q \rangle \in red \wedge \langle q, r, s \rangle \in andI) \Rightarrow [r/p]|P| \wedge [s/p]|Q|)$,
- $|P \vee Q| = p \in sn \wedge \forall q\ \forall r\ ((\langle p, q \rangle \in red \wedge \langle q, r \rangle \in orI1) \Rightarrow [r/p]|P|) \wedge \forall q\ \forall r\ ((\langle p, q \rangle \in red \wedge \langle q, r \rangle \in orI2) \Rightarrow [r/p]|Q|)$,
- $|\bot| = p \in sn$,
- $|\forall x\ P| = p \in sn \wedge \forall q\ \forall w\ \forall r\ (\langle p, q \rangle \in red \wedge (\langle q, w, r \rangle \in forallI) \Rightarrow \forall x\ \forall y\ (x \in E_{S(x)} \wedge y \in Term) \Rightarrow \forall s\ (\langle s, w, y, r \rangle \in subst' \Rightarrow [r/p, x/x]|P|))$,
- $|\exists x\ P| = p \in sn \wedge \forall q\ \forall t\ \forall r\ (\langle p, q \rangle \in red \wedge (\langle q, t, r \rangle \in existsI) \Rightarrow \exists x\ x \in E_{S(x)} \Rightarrow [r/p, x/x]|P|))$.

Notice that the free variables of $|P|$ are among $term, subst, subst', red, sn, impI, andI, orI1, orI2, forallI, existsI,\ p, E_0, \ldots, E_m, x_1, \ldots, x_n, x_{n+1}$.

We let

$$\varphi = a_1/x_1, \ldots, a_n/x_n, a_{n+1}/x_{n+1}$$

$$\psi = Term/term, Subst/subst, Subst'/subst', Red/red, Sn/sn,$$

$$ImpI/impI, AndI/andI, OrI1/orI1, OrI2/orI2, ForallI/forallI, ExistsI/existsI,$$

$$A_0/E_0, \ldots, A_m/E_m, \tau^{k_1} a_1/x_1, \ldots, \tau^{k_n} a_n/x_n, \tau^{k_{n+1}} a_{n+1}/x_{n+1}$$

We check, by induction over the structure of P, that if P is a stratifiable proposition, then the set of proofs π such that $[\![|P|]\!]^\mathcal{M}_{\psi + \pi/p} = 1$ is $[P]_\varphi$.

- If P is an atomic proposition $x_i \in x_j$, then $k_j = k_i + 1$, we have $[\![|P|]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if $\langle \pi, \tau^{k_i} a_i \rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau^{k_j} a_j$ if and only if $\langle \tau^{k_i}\pi, \tau^{k_i} a_i \rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau^{k_j} a_j$ if and only if $\tau^{k_i}\langle \pi, a_i \rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau^{k_j} a_j$ if and only if $\langle \pi, a_i \rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau a_j$ if and only if π is in $[P]_\varphi$.

- if $P = Q \Rightarrow R$ then we have $[\![|P|]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\lambda\alpha\ \pi_1$ then for all π' such that $[\![|Q|]\!]^{\mathcal{M}}_{\psi+\pi'/p} = 1$ we have $[\![|R|]\!]^{\mathcal{M}}_{\psi+[\pi'/\alpha]\pi_1/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\lambda x\ \pi_1$ then for all π' in $[Q]_\varphi$, $[\pi'/\alpha]\pi_1$ is in $[R]_\varphi$ if and only if π is in $[P]_\varphi$.

- If $P = Q \wedge R$ then we have $[\![P]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\langle \pi_1, \pi_2 \rangle$ then $[\![Q]\!]^{\mathcal{M}}_{\psi+\pi_1/p} = 1$ and $[\![R]\!]^{\mathcal{M}}_{\psi+\pi_2/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\langle \pi_1, \pi_2 \rangle$ then π_1 is in $[Q]_\varphi$ and π_2 is in $[R]_\varphi$ if and only if π is in $[P]_\varphi$.

- If $P = Q \vee R$ then we have $[\![P]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $i(\pi_1)$ (resp. $j(\pi_2)$) then $[\![P]\!]^{\mathcal{M}}_{\psi+\pi_1/p} = 1$ (resp. $[\![Q]\!]^{\mathcal{M}}_{\psi+\pi_2/p} = 1$) if and only if π is strongly normalizable and whenever π reduces to $i(\pi_1)$ (resp. $j(\pi_2)$) then π_1 is in $[P]_\varphi$ (resp. $[Q]_\varphi$) if and only if π is in $[P]_\varphi$.

- If $P = \bot$ then $[\![P]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable if and only if π is in $[P]_\varphi$.

- if $P = \forall x\ Q$, then $[\![|P|]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\lambda x\ \pi_1$, for all term t and for all c in M such that $c \in_{\mathcal{M}} A_k$, $[\![|Q|]\!]^{\mathcal{M}}_{\psi+c/x,[t/x]\pi_1/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\lambda x\ \pi_1$, for all t and for all e in N $[\![|Q|]\!]^{\mathcal{M}}_{\psi+\tau^k e/x+[t/x]\pi_1/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\lambda x\ \pi_1$, for all t and for all e in N, $[t/x]\pi_1 \in [Q]_{\varphi+e/x}$ if and only if π is in $[P]_\varphi$.

- if $P = \exists x\ Q$, then $[\![|P|]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\langle t, \pi_1 \rangle$, there exists a c in M such that $c \in_{\mathcal{M}} A_k$ and $[\![|Q|]\!]^{\mathcal{M}}_{\psi+c/x,[t/x]\pi_1/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\langle t, \pi_1 \rangle$, there exists a e in N such that $[\![|Q|]\!]^{\mathcal{M}}_{\psi+\tau^k e/x+[t/x]\pi_1/p} = 1$ if and only if π is strongly normalizable and whenever π reduces to $\langle t, \pi_1 \rangle$, there exists a e in N, $[t/x]\pi_1 \in [Q]_{\varphi+e/x}$ if and only if π is in $[P]_\varphi$.

Then, the model $\mathcal{M}$ is a model of the comprehension scheme. Hence, it is a model of the proposition

$$\forall E_0 \ldots \forall E_m \forall x_1 \ldots \forall x_n \exists z \forall p \forall x_{n+1} \langle p, x_{n+1}\rangle \in z \Leftrightarrow \langle p, x_{n+1}\rangle \in \mathbb{N}\times U \wedge |P|$$

i.e.

$$[\![\forall E_0 \ldots \forall E_m \forall x_1 \ldots \forall x_n \exists z \forall p \forall x_{n+1} \langle p, x_{n+1}\rangle \in z \Leftrightarrow \langle p, x_{n+1}\rangle \in \mathbb{N}\times U \wedge |P|]\!]^{\mathcal{M}} = 1$$

Thus there exists a b_0 such that

$$[\![\langle p, x_{n+1}\rangle \in z \Leftrightarrow \langle p, x_{n+1}\rangle \in \mathbb{N}_{\mathcal{M}} \times U \wedge |P|]\!]^{\mathcal{M}}_{\psi+b_0/z, u_{k_{n+1}+1}/U, \pi/p} = 1$$

We have $\langle \pi, \tau^{k_{n+1}} a_{n+1}\rangle_{\mathcal{M}} \in_{\mathcal{M}} b_0$ if and only if $\tau^{k_{n+1}} a_{n+1} \in_{\mathcal{M}} u_{k_{n+1}+1}$ and $[\![|P|]\!]^{\mathcal{M}}_{\psi+\pi/p} = 1$. Thus $\langle \pi, a_{n+1}\rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau^{k_{n+1}} b_0$ if and only if $a_{n+1} \in_{\mathcal{M}} u_1$ and π is in $[P]_{\varphi}$.

We take $b = \tau^{-(k_{n+1}+1)} b_0$ and for all a_{n+1} in N we have $\langle \pi, a_{n+1}\rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau b$ if and only if π is in $[P]_{\varphi}$.

Finally, notice that b_0 is a set of pairs $\langle \pi, \beta\rangle_{\mathcal{M}}$ where π is a proof and β an element of $u_{k_{n+1}+1}$ and for each β in $u_{k_{n+1}+1}$ the set of π such that $\langle \pi, \beta\rangle_{\mathcal{M}} \in_{\mathcal{M}} b_0$ is $[\![|P|]\!]^{\mathcal{M}}_{\psi+\beta/x_{k_{n+1}}, \pi/p} = 1$, hence it is a reducibility candidate. Hence $b_0 \in_{\mathcal{M}} A_{k_{n+1}+1}$ and b is in N. □

Definition 2.9 (Crabbé's pre-model) *The pre-model $\mathcal{N} = \langle N, \in_{\mathcal{N}}, \hat{f}_{x_1,\ldots,x_n,y,P}\rangle$ is defined as follows. The base set is N. If α and β are elements of N we take $\in_{\mathcal{N}} (\alpha, \beta) = \{\pi \mid \langle \pi, \alpha\rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau\beta\}$.*

The function $\hat{f}_{x_1,\ldots,x_n,x_{n+1},P}$ maps $(a_1, \ldots, a_n)$ to the object b such that for all a_{n+1} in N, $\langle \pi, a_{n+1}\rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau b$ if and only if π is in $[P]_{a_1/x_1,\ldots,a_n/x_n,a_{n+1}/x_{n+1}}$.

Proposition 2.9 *The pre-model $\mathcal{N}$ is a pre-model of the Stratified Foundations.*

Proof. By induction over the structure of P, if P is a proposition in the language $\in$, its denotation in $\mathcal{N}$ for the assignment φ is $[P]_{\varphi}$. Then π is in $[\![t_{n+1} \in f_{x_1,\ldots,x_n,x_{n+1},P}(t_1, \ldots, t_n)]\!]^{\mathcal{N}}_{\varphi}$ if and only if

$$\langle \pi, [\![t_{n+1}]\!]^{\mathcal{N}}_{\varphi}\rangle_{\mathcal{M}} \in_{\mathcal{M}} \tau \hat{f}_{x_1,\ldots,x_n,x_{n+1},P}([\![t_1]\!]^{\mathcal{N}}_{\varphi}, \ldots, [\![t_n]\!]^{\mathcal{N}}_{\varphi})$$

if and only if π is in $[P]_{[\![t_1]\!]^{\mathcal{N}}_{\varphi}/x_1,\ldots,[\![t_n]\!]^{\mathcal{N}}_{\varphi}/x_n,[\![t_{n+1}]\!]^{\mathcal{N}}_{\varphi}/x_{n+1}}$ if and only if π is in $[\![P]\!]^{\mathcal{U}}_{[\![t_1]\!]^{\mathcal{N}}_{\varphi}/x_1,\ldots,[\![t_n]\!]^{\mathcal{N}}_{\varphi}/x_n,[\![t_{n+1}]\!]^{\mathcal{N}}_{\varphi}/x_{n+1}}$ if and only if π is in $[\![[t_1/x_1, \ldots, t_n/x_n, t_{n+1}/x_{n+1}]P]\!]^{\mathcal{N}}_{\varphi}$.

Hence, if $A \equiv B$ then A and B have the same denotation. □

Corollary 2.1 *All proofs in the Stratified Foundations strongly normalize.*

Conclusion

In this note, we have have shown that the Stratified Foundations can be expressed in deduction modulo and that the normalization proof for this theory be decomposed into two lemmas: one expressing that it has a pre-model and the other that proof normalize in this theory if it has a pre-model. This second lemma is not specific to the Stratified Foundations, but holds for all theories modulo.

The first lemma does not seem to be specific either. Indeed, as noticed by Crabbé, the model $\mathcal{M}$ could be replaced by a weakly extensional ω-model of the Stratified Foundations. The idea of this normalization proof is hence to construct a pre-model within an ω-model of some theory with the help of formal realizability. The generality of this idea reamains to be investigated. Thus, this example contributes to explore of the border between the theories modulo that have the normalization property and those that do not.

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