The modal multilogic of geometry

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Abstract

A spatial logic is a modal logic of which the models are the mathematical models of space. Successively considering the mathematical models of space that are the incidence geometry and the projective geometry, we will successively establish the language, the semantical basis, the axiomatical presentation, the proof of the decidability and the proof of the completeness of INC, the modal multilogic of incidence geometry, and PRO, the modal multilogic of projective geometry.

1 Introduction

Our perception of space is less direct than our perception of time which is not the result of thought but the outcome of consciousness. Nevertheless, we should acknowledge that space more vigorously asserts its truth to our senses than time : we can see the objects that occupy space whereas we cannot know similarly the ways of the events that fill time [12].

Well then, why, since the works of Dummett and Lemmon [8], Hintikka [14] and Prior [22] on the logic of the Diodorean modalities, has temporal logic become the industry of today (specification and verification of programs and systems [1], applications in natural language, concurrent computation, planning and databases [11], logics and semantics of programming [18], etcetera) whereas spatial logic is still in its infancy ? Because, in actual fact, there are but a few modal logics which are for space what temporal logics are for time. And even are they presented without a common line of action, with the result that we cannot consider working out a general theory of spatial logic.

The relevance of the modal logics designed by Balbiani, Fariñas del Cerro, Tinchev and Vakarelov [4], Bennett [5] and Jeansoulin and Mathieu [17] for spatial reasoning is debatable. On the one hand, it is just that Balbiani, Fariñas del Cerro, Tinchev and Vakarelov regard the arrow frame à la Vakarelov [25] as a mathematical model of space : their choice leads to a too abstract modal logic of space. On the other hand, it is just that Bennett considers a topological interpretation of the intuitionistic propositional calculus and that Jeansoulin and Mathieu look upon $S4$ as the modal logic of inclusion : there is nothing original about their propositions.

By the way, what is the measure by which we can judge whether or not a modal logic is a spatial logic ? For several years, a temporal logic is a modal logic of which the models are the mathematical models of time [10]. By analogy with temporal logic, in agreement with Lemon [19], we will say that a spatial logic is a modal logic of which the models are the mathematical models of space. A mathematical model of space is a relational structure consisting of one or more sets of geometrical beings (*lines*, *points*, etcetera) together with one or more basic relations between these geometrical beings (*incidence* between *lines* and *points*, *parallelism* between *lines*, *orthogonality* between *lines*, etcetera). The simplest mathematical model of space is the frame of incidence that is a relational structure of the form $\mathcal{F} = (Li, Po, in)$ where Li is a nonempty set of geometrical beings of type *line*, Po is a nonempty set of geometrical beings of type *point* and in is a binary relation of *incidence* between *points* and *lines* such that two distinct *points* are together *incident* with exactly one *line*.

Traditionnally, the semantical basis of modal logic is a relational structure consisting of one set of possible worlds together with one relation between these possible worlds [15]. The possible applications of modal logic (reasoning about knowledge [9], reasoning about programs [13], reasoning about objects [21], etcetera) have given prominence to relational structures consisting of one set of possible worlds together with several relations between these possible worlds. It is only lately that, in the context of dynamic arrow logic, van Benthem [6], Marx [20] and de Rijke [24] have resolved to consider relational structures consisting of several sets of possible worlds together with several relations between these possible worlds.

The frame of incidence is a relational structure consisting of two sets of geometrical beings together with one relation between these geometrical beings. Backing up our reflection with the achievements of van Benthem, Marx and de Rijke and successively considering the mathematical models of space that are the incidence geometry and the projective geometry (that is an incidence geometry in which two distinct *lines* are together *incident* with exactly one *point*), we will successively establish the language, the semantical basis, the axiomatical presentation, the proof of the decidability and the proof of the completeness of INC, the modal multilogic of incidence geometry, and PRO, the modal multilogic of projective geometry. The proof of the decidability of the modal multilogic of incidence geometry and the proof of the decidability of the modal multilogic of projective geometry use the techniques of the filtration introduced by Segerberg [15]. The proof of the completeness of the modal multilogic of incidence geometry and the proof of the completeness of the modal multilogic of projective geometry use the techniques of the frame of subordination introduced by Cresswell [15] and developed by Balbiani [2] [3] and Humberstone [16].

2 Language

The linguistic basis of the modal multilogic of geometry is the propositional calculus. Let LIN be a nonempty set of atomic formulas of type *line* and POI be a nonempty set of atomic formulas of type *point*. The set $FORLIN$ of the complex formulas of type *line* and the set $FORPOI$ of the complex formulas of type *point* are defined by induction in the following way :

- $LIN \subseteq FORLIN$.
- $POI \subseteq FORPOI$.
- For every $\alpha, \beta \in FORLIN$, $\alpha \vee \beta \in FORLIN$.
- For every $A, B \in FORPOI$, $A \vee B \in FORPOI$.
- For every $\alpha \in FORLIN$, $\neg\alpha \in FORLIN$.
- For every $A \in FORPOI$, $\neg A \in FORPOI$.
- For every $A \in FORPOI$, $[on]A \in FORLIN$.
- For every $\alpha \in FORLIN$, $[in]\alpha \in FORPOI$.

For every $A \in FORPOI$, let $\langle on \rangle A = \neg[on]\neg A$. For every $\alpha \in FORLIN$, let $\langle in \rangle \alpha = \neg[in]\neg\alpha$.

The frame of incidence is a relational structure consisting of two sets of geometrical beings together with one relation between these geometrical beings. Consequently, it is only natural that the decision should have been reached to consider a language made up of two sets of formulas together with the modal operators $[on]$ and $[in]$ permitting to go from one set to another in the following way :

- For every $A \in FORPOI$, the complex formula $[on]A$ of type *line* signifies "in every *point incident* with the current *line*, it is the case that A".
- For every $\alpha \in FORLIN$, the complex formula $[in]\alpha$ of type *point* signifies "in every *line incident* with the current *point*, it is the case that α".

Consequently :

- For every $A \in FORPOI$, the complex formula $\langle on \rangle A$ signifies "in some *point incident* with the current *line*, it is the case that A".
- For every $\alpha \in FORLIN$, the complex formula $\langle in \rangle \alpha$ signifies "in some *line incident* with the current *point*, it is the case that α".
- For every $\alpha \in FORLIN$, the complex formula $[on][in]\alpha$ signifies "in every *line incident* with any *point incident* with the current *line*, it is the case that α".
- For every $A \in FORPOI$, the complex formula $[in][on]A$ signifies "in every *point incident* with any *line incident* with the current *point*, it is the case that A".

The modalities of type *line* to *line* and the modalities of type *point* to *point* are defined by induction in the following way :

- The empty modality is a modality of type *line* to *line*.
- The empty modality is a modality of type *point* to *point*.
- For every modality λ of type *line* to *line*, $[on][in]\lambda$ is a modality of type *line* to *line*.
- For every modality λ of type *point* to *point*, $[in][on]\lambda$ is a modality of type *point* to *point*.

3 Semantical study

This section presents the semantical study of the modal multilogic of geometry.

3.1 Basic frame

A basic frame is a relational structure of the form $\mathcal{F} = (Li, Po, on, in)$ where Li is a nonempty set of geometrical beings of type *line*, Po is a nonempty set of geometrical beings of type *point*, on is a binary relation of *incidence* on Li and Po and in is a binary relation of *incidence* on Po and Li such that :

- For every $x \in Li$, $on(x) \neq \emptyset$ (every *line* is *incident* with at least one *point*).
- For every $X \in Po$, $in(X) \neq \emptyset$ (every *point* is *incident* with at least one *line*).
- For every $x \in Li$ and for every $X \in Po$, if $X \in on(x)$ then $x \in in(X)$ (if a *line* is *incident* with a *point* then the *point* is *incident* with the *line*).
- For every $X \in Po$ and for every $x \in Li$, if $x \in in(X)$ then $X \in on(x)$ (if a *point* is *incident* with a *line* then the *line* is *incident* with the *point*).

$\mathcal{F}$ is normal when :

- For every $X, Y \in Po$ and for every $x, y \in Li$, if $\{x, y\} \subseteq in(X) \cap in(Y)$ then $X = Y$ or $x = y$ (if two *points* are *incident* with two *lines* then either the two *points* are equal or the two *lines* are equal).

Direct calculations would lead to the conclusion that :

Proposition 1 *If $\mathcal{F}$ is normal then :*

- *For every $x, y \in Li$ and for every $X, Y \in Po$, if $\{X, Y\} \subseteq on(x) \cap on(y)$ then $x = y$ or $X = Y$ (if two* lines *are* incident *with two* points *then either the two* lines *are equal or the two* points *are equal).*

$\mathcal{F}$ is connected when :

- For every $X, Y \in Po$, there exists $k \geq 0$ and there exists $X_0, \ldots, X_k \in Po$ such that :
  - $X_0 = X$.
  - For every $l \in \{1, \ldots, k\}$, $in(X_{l-1}) \cap in(X_l) \neq \emptyset$.
  - $X_k = Y$.

The reader may easily verify that :

Proposition 2 *If $\mathcal{F}$ is connected then :*

- *For every $x, y \in Li$, there exists $k \geq 0$ and there exists $x_0, \ldots, x_k \in Li$ such that :*
  - *$x_0 = x$.*
  - *For every $l \in \{1, \ldots, k\}$, $on(x_{l-1}) \cap on(x_l) \neq \emptyset$.*
  - *$x_k = y$.*

$\mathcal{F}$ is a frame of incidence when :

- For every $X, Y \in Po$, $in(X) \cap in(Y) \neq \emptyset$ (two *points* are together *incident* with at least one *line*).

$\mathcal{F}$ is a projective frame when :

- For every $x, y \in Li$, $on(x) \cap on(y) \neq \emptyset$ (two *lines* are together *incident* with at least one *point*).
- For every $X, Y \in Po$, $in(X) \cap in(Y) \neq \emptyset$.

It is easy to verify that :

Proposition 3 *If $\mathcal{F}$ is a frame of incidence then $\mathcal{F}$ is connected.*

Proposition 4 *If $\mathcal{F}$ is a projective frame then $\mathcal{F}$ is connected.*

Moreover :

Proposition 5 *If $\mathcal{F}$ is a normal frame of incidence then, for every $X, Y \in Po$, if $X \neq Y$ then $Card(in(X) \cap in(Y)) = 1$ (two distinct* points *are together* incident *with exactly one* line*).*

Proposition 6 *If $\mathcal{F}$ is a normal projective frame then, for every $x, y \in Li$, if $x \neq y$ then $Card(on(x) \cap on(y)) = 1$ (two distinct* lines *are together* incident *with exactly one* point*). Moreover, for every $X, Y \in Po$, if $X \neq Y$ then $Card(in(X) \cap in(Y)) = 1$.*

3.2 Valuation

Let $\mathcal{F} = (Li, Po, on, in)$ be a basic frame. A valuation on $\mathcal{F}$ is a structure of the form (R, V) where R is a mapping of LIN to $\mathcal{P}(Li)$ and V is a mapping of POI to $\mathcal{P}(Po)$. The mapping $\widetilde{R}$ of $FORLIN$ to $\mathcal{P}(Li)$ and the mapping $\widetilde{V}$ of $FORPOI$ to $\mathcal{P}(Po)$ are defined by induction in the following way :

- For every $\pi \in LIN$, $\widetilde{R}(\pi) = R(\pi)$.
- For every $p \in POI$, $\widetilde{V}(p) = V(p)$.
- For every $\alpha, \beta \in FORLIN$, $\widetilde{R}(\alpha \vee \beta) = \widetilde{R}(\alpha) \cup \widetilde{R}(\beta)$.
- For every $A, B \in FORPOI$, $\widetilde{V}(A \vee B) = \widetilde{V}(A) \cup \widetilde{V}(B)$.
- For every $\alpha \in FORLIN$, $\widetilde{R}(\neg\alpha) = Li \setminus \widetilde{R}(\alpha)$.
- For every $A \in FORPOI$, $\widetilde{V}(\neg A) = Po \setminus \widetilde{V}(A)$.
- For every $A \in FORPOI$, $\widetilde{R}([on]A) = \{x : on(x) \subseteq \widetilde{V}(A)\}$.
- For every $\alpha \in FORLIN$, $\widetilde{V}([in]\alpha) = \{X : in(X) \subseteq \widetilde{R}(\alpha)\}$.

Our definition yields the following result :

Proposition 7 *For every $A \in FORPOI$, $\widetilde{R}(\langle on\rangle A) = \{x : on(x) \cap \widetilde{V}(A) \neq \emptyset\}$. Moreover, for every $\alpha \in FORLIN$, $\widetilde{V}(\langle in\rangle\alpha) = \{X : in(X) \cap \widetilde{R}(\alpha) \neq \emptyset\}$.*

Moreover :

Proposition 8 *If $\mathcal{F}$ is a frame of incidence then, for every $A \in FORPOI$, if $\widetilde{V}(A) = Po$ then $\widetilde{V}([in][on]A) = Po$ else $\widetilde{V}([in][on]A) = \emptyset$.*

Proof If $\widetilde{V}([in][on]A) \neq \emptyset$ then there exists $X \in Po$ such that $in(X) \subseteq \widetilde{R}([on]A)$. Consequently, for every $x \in Li$, if $x \in in(X)$ then $on(x) \subseteq \widetilde{V}(A)$. Since $\mathcal{F}$ is a frame of incidence, then, for every $Y \in Po$, there is $x \in Li$ such that $x \in in(X)$ and $Y \in on(x)$. Consequently, for every $Y \in Po$, $Y \in \widetilde{V}(A)$. Consequently, $\widetilde{V}(A) = Po$ and $\widetilde{V}([in][on]A) = Po$.
$\dashv$

Proposition 9 *If $\mathcal{F}$ is a projective frame then, for every $\alpha \in FORLIN$, if $\widetilde{R}(\alpha) = Li$ then $\widetilde{R}([on][in]\alpha) = Li$ else $\widetilde{R}([on][in]\alpha) = \emptyset$. Moreover, for every $A \in FORPOI$, if $\widetilde{V}(A) = Po$ then $\widetilde{V}([in][on]A) = Po$ else $\widetilde{V}([in][on]A) = \emptyset$.*

Proof If $\widetilde{R}([on][in]\alpha) \neq \emptyset$ then there exists $x \in Li$ such that $on(x) \subseteq \widetilde{V}([in]\alpha)$. Consequently, for every $X \in Po$, if $X \in on(x)$ then $in(X) \subseteq \widetilde{R}(\alpha)$. Since $\mathcal{F}$ is a projective frame, then, for every $y \in Li$, there is $X \in Po$ such that $X \in on(x)$ and $y \in in(X)$. Consequently, for every $y \in Li$, $y \in \widetilde{R}(\alpha)$. Consequently, $\widetilde{R}(\alpha) = Li$ and $\widetilde{R}([on][in]\alpha) = Li$.
$\dashv$

3.3 Basic model

Let $\mathcal{F} = (Li, Po, on, in)$ be a basic frame and (R, V) be a valuation on $\mathcal{F}$. The structure $\mathcal{M} = (Li, Po, on, in, R, V)$ is called basic model on $\mathcal{F}$ defined from (R, V). Let ... :

- ... K be the class of all basic models and K^* be the class of all countable, normal and connected basic models.
- ... INC be the class of all models of incidence and INC^* be the class of all countable and normal models of incidence.
- ... PRO be the class of all projective models and PRO^* be the class of all countable and normal projective models.

Let ... :

- ... K° be the class of all connected basic models and K°_f be the class of all finite and connected basic models.
- ... INC_f be the class of all finite models of incidence.
- ... PRO_f be the class of all finite projective models.

The relation of satisfiability in $\mathcal{M}$ of a formula is defined in the following way :

- For every $x \in Li$ and for every $\alpha \in FORLIN$, $x \models_{\mathcal{M}} \alpha$ iff $x \in \widetilde{R}(\alpha)$.
- For every $X \in Po$ and for every $A \in FORPOI$, $X \models_{\mathcal{M}} A$ iff $X \in \widetilde{V}(A)$.

The relation of validity in $\mathcal{M}$ of a formula is defined in the following way :

- For every $\alpha \in FORLIN$, $\models_{\mathcal{M}} \alpha$ iff, for every $x \in Li$, $x \models_{\mathcal{M}} \alpha$.
- For every $A \in FORPOI$, $\models_{\mathcal{M}} A$ iff, for every $X \in Po$, $X \models_{\mathcal{M}} A$.

Let $\mathcal{K}$ be a class of models. The relation of validity in $\mathcal{K}$ of a formula is defined in the following way :

- For every $\alpha \in FORLIN$, $\models_{\mathcal{K}} \alpha$ iff, for every $\mathcal{M} \in \mathcal{K}$, $\models_{\mathcal{M}} \alpha$.
- For every $A \in FORPOI$, $\models_{\mathcal{K}} A$ iff, for every $\mathcal{M} \in \mathcal{K}$, $\models_{\mathcal{M}} A$.

Theorem 1 *Let $\mathcal{M} = (Li, Po, on, in, R, V)$ be a basic model. For every $x \in Li$ and for every $\alpha \in FORLIN$, $x \models_{\mathcal{M}} \alpha$ iff $x \models_{\mathcal{M}^x} \alpha$ ($\mathcal{M}^x$ being the connected submodel of $\mathcal{M}$ containing x). Moreover, for every $X \in Po$ and for every $A \in FORPRO$, $X \models_{\mathcal{M}} A$ iff $X \models_{\mathcal{M}^X} A$ ($\mathcal{M}^X$ being the connected submodel of $\mathcal{M}$ containing X).*

Proof By induction on the complexity of α and on the complexity of A.
$\dashv$
Consequently :

Theorem 2 *For every $\alpha \in FORLIN$, if $\models_{K^\circ} \alpha$ then $\models_K \alpha$. Moreover, for every $A \in FORPOI$, if $\models_{K^\circ} A$ then $\models_K A$.*

Proof If $\not\models_K \alpha$ then there exists $\mathcal{M} = (Li, Po, on, in, R, V) \in K$ such that $\not\models_{\mathcal{M}} \alpha$. Consequently, there exists $x \in Li$ such that $x \not\models_{\mathcal{M}} \alpha$. According to the theorem 1, $x \not\models_{\mathcal{M}^x} \alpha$ ($\mathcal{M}^x$ being the connected submodel of $\mathcal{M}$ containing x). Consequently, $\not\models_{\mathcal{M}^x} \alpha$. Direct calculations would lead to the conclusion that $\mathcal{M}^x \in K^\circ$. Consequently, $\not\models_{K^\circ} \alpha$.
$\dashv$

3.4 Filtration

A filter is a structure of the form (Λ, Π) where Λ is a subset of $FORLIN$ and Π is a subset of $FORPOI$ such that :

- For every $\alpha, \beta \in FORLIN$, if $\alpha \vee \beta \in \Lambda$ then $\alpha \in \Lambda$ and $\beta \in \Lambda$.
- For every $A, B \in FORPOI$, if $A \vee B \in \Pi$ then $A \in \Pi$ and $B \in \Pi$.
- For every $\alpha \in FORLIN$, if $\neg\alpha \in \Lambda$ then $\alpha \in \Lambda$.
- For every $A \in FORPOI$, if $\neg A \in \Pi$ then $A \in \Pi$.
- For every $A \in FORPOI$, if $[on]A \in \Lambda$ then $A \in \Pi$.
- For every $\alpha \in FORLIN$, if $[in]\alpha \in \Pi$ then $\alpha \in \Lambda$.

Let (Λ, Π) be a filter and $\mathcal{M} = (Li, Po, on, in, R, V)$ be a basic model. Let $\equiv_\Lambda$ be the relation of equivalence on Li defined in the following way :

- For every $x, y \in Li$, $x \equiv_\Lambda y$ iff, for every $\alpha \in \Lambda$, $x \in \widetilde{R}(\alpha)$ iff $y \in \widetilde{R}(\alpha)$.

and $\equiv_\Pi$ be the relation of equivalence on Po defined in the following way :

- For every $X, Y \in Po$, $X \equiv_\Pi Y$ iff, for every $A \in \Pi$, $X \in \widetilde{V}(A)$ iff $Y \in \widetilde{V}(A)$.

Let $Li' = Li_{|\equiv_\Lambda}$ and $Po' = Po_{|\equiv_\Pi}$. Let on' be a binary relation on Li' and Po' such that :

- For every $x \in Li$ and for every $X \in Po$, if $X \in on(x)$ then $\equiv_\Pi (X) \in on'(\equiv_\Lambda (x))$.
- For every $x \in Li$ and for every $X \in Po$, if $\equiv_\Pi (X) \in on'(\equiv_\Lambda (x))$ then, for every $A \in FORPOI$, if $[on]A \in \Lambda$ and $x \in \widetilde{R}([on]A)$ then $X \in \widetilde{V}(A)$.

and in' be a binary relation on Po' and Li' such that :

- For every $X \in Po$ and for every $x \in Li$, if $x \in in(X)$ then $\equiv_\Lambda (x) \in in'(\equiv_\Pi (X))$.
- For every $X \in Po$ and for every $x \in Li$, if $\equiv_\Lambda (x) \in in'(\equiv_\Pi (X))$ then, for every $\alpha \in FORLIN$, if $[in]\alpha \in \Pi$ and $X \in \widetilde{V}([in]\alpha)$ then $x \in \widetilde{R}(\alpha)$.

Let R' be a mapping of LIN to $\mathcal{P}(Li')$ such that :

- For every $\pi \in LIN$, if $\pi \in \Lambda$ then $R'(\pi) = \{\equiv_\Lambda (x) : x \in R(\pi)\}$.

and V' be a mapping of POI to $\mathcal{P}(Po')$ such that :

- For every $p \in POI$, if $p \in \Pi$ then $V'(p) = \{\equiv_\Pi (X) : X \in V(p)\}$.

The structure $\mathcal{M}' = (Li', Po', on', in', R', V')$ is called filtration of $\mathcal{M}$ through (Λ, Π).

Proposition 10 *For every $\alpha \in FORLIN$, if $\alpha \in \Lambda$ then $\widetilde{R'}(\alpha) = \{\equiv_\Lambda (x) : x \in \widetilde{R}(\alpha)\}$. Moreover, for every $A \in FORPOI$, if $A \in \Pi$ then $\widetilde{V'}(A) = \{\equiv_\Pi (X) : X \in \widetilde{V}(A)\}$.*

Proof By induction on the complexity of α and on the complexity of A.
$\dashv$
Consequently :

Theorem 3 *For every $\alpha \in FORLIN$, if $\models_{K_f^\circ} \alpha$ then $\models_{K^\circ} \alpha$. Moreover, for every $A \in FORPOI$, if $\models_{K_f^\circ} A$ then $\models_{K^\circ} A$.*

Proof If $\not\models_{K^\circ} \alpha$ then there exists $\mathcal{M} = (Li, Po, on, in, R, V) \in K^\circ$ such that $\not\models_{\mathcal{M}} \alpha$. Consequently, there exists $x \in Li$ such that $x \not\models_{\mathcal{M}} \alpha$. Consequently, $x \notin \widetilde{R}(\alpha)$. Let (Λ, Π) be a finite filter such that $\alpha \in \Lambda$ and $\mathcal{M}' = (Li', Po', on', in', R', V')$ be the filtration of $\mathcal{M}$ through (Λ, Π) defined in the following way :

- For every $x \in Li$ and for every $X \in Po$, $\equiv_\Pi (X) \in on'(\equiv_\Lambda (x))$ iff there exists $y \in Li$ and there exists $Y \in Po$ such that $x \equiv_\Lambda y$, $X \equiv_\Pi Y$ and $Y \in on(y)$.
- For every $X \in Po$ and for every $x \in Li$, $\equiv_\Lambda (x) \in in'(\equiv_\Pi (X))$ iff there exists $Y \in Po$ and there exists $y \in Li$ such that $X \equiv_\Pi Y$, $x \equiv_\Lambda y$ and $y \in in(Y)$.

Direct calculations would lead to the conclusion that $\mathcal{M}' \in K_f^\circ$. According to the proposition 10, $\equiv_\Lambda (x) \notin \widetilde{R'}(\alpha)$. Consequently, $\equiv_\Lambda (x) \not\models_{\mathcal{M}'} \alpha$, $\not\models_{\mathcal{M}'} \alpha$ and $\not\models_{K_f^\circ} \alpha$.
$\dashv$

Moreover :

Theorem 4 *Let $\mathcal{L} \in \{INC, PRO\}$. For every $\alpha \in FORLIN$, if $\models_{\mathcal{L}_f} \alpha$ then $\models_{\mathcal{L}} \alpha$. Moreover, for every $A \in FORPOI$, if $\models_{\mathcal{L}_f} A$ then $\models_{\mathcal{L}} A$.*

Proof If $\not\models_{\mathcal{L}} \alpha$ then there exists $\mathcal{M} = (Li, Po, on, in, R, V) \in \mathcal{L}$ such that $\not\models_{\mathcal{M}} \alpha$. Consequently, there exists $x \in Li$ such that $x \not\models_{\mathcal{M}} \alpha$. Consequently, $x \notin \widetilde{R}(\alpha)$. Let (Λ, Π) be a finite filter such that $\alpha \in \Lambda$ and $\mathcal{M}' = (Li', Po', on', in', R', V')$ be the filtration of $\mathcal{M}$ through (Λ, Π) defined in the following way :

- For every $x \in Li$ and for every $X \in Po$, $\equiv_\Pi (X) \in on'(\equiv_\Lambda (x))$ iff there exists $y \in Li$ and there exists $Y \in Po$ such that $x \equiv_\Lambda y$, $X \equiv_\Pi Y$ and $Y \in on(y)$.
- For every $X \in Po$ and for every $x \in Li$, $\equiv_\Lambda (x) \in in'(\equiv_\Pi (X))$ iff there exists $Y \in Po$ and there exists $y \in Li$ such that $X \equiv_\Pi Y$, $x \equiv_\Lambda y$ and $y \in in(Y)$.

Direct calculations would lead to the conclusion that $\mathcal{M}' \in \mathcal{L}_f$. According to the proposition 10, $\equiv_\Lambda (x) \notin \widetilde{R'}(\alpha)$. Consequently, $\equiv_\Lambda (x) \not\models_{\mathcal{M}'} \alpha$, $\not\models_{\mathcal{M}'} \alpha$ and $\not\models_{\mathcal{L}_f} \alpha$.
$\dashv$

4 Axiomatical study

This section presents the axiomatical study of the modal multilogic of geometry.

4.1 Axiomatical presentation of K

Together with the classical tautologies, all the instances of the following schemata :

- $[on](A \to B) \to ([on]A \to [on]B)$.
- $[in](\alpha \to \beta) \to ([in]\alpha \to [in]\beta)$.

and all the instances of the following schemata :

- $[on]A \to \langle on \rangle A$.
- $[in]\alpha \to \langle in \rangle \alpha$.
- $\alpha \to [on]\langle in \rangle \alpha$.
- $A \to [in]\langle on \rangle A$.

are axioms of K. Together with the classical rules of inference, all the instances of the following schemata :

- If A is a theorem then $[on]A$ is a theorem.
- If α is a theorem then $[in]\alpha$ is a theorem.

are rules of inference of K.

4.2 Axiomatical presentation of INC

Together with the axioms and the rules of inference of K, all the instances of the following schema :

- $[in]\alpha \to \lambda\langle in \rangle \alpha$, for every modality λ of type *point* to *point*.

are axioms of INC, the modal multilogic of incidence geometry.

4.3 Axiomatical presentation of PRO

Together with the axioms and the rules of inference of K, all the instances of the following schemata :

- $[on]A \to \lambda\langle on \rangle A$, for every modality λ of type *line* to *line*.

- $[in]\alpha \rightarrow \lambda\langle in\rangle\alpha$, for every modality λ of type *point* to *point*.

are axioms of PRO, the modal multilogic of projective geometry.

4.4 Soundness

Theorem 5 *Let $\mathcal{L} \in \{K, INC, PRO\}$. For every $\alpha \in FORLIN$, if α is a theorem of $\mathcal{L}$ then $\models_{\mathcal{L}} \alpha$. Moreover, for every $A \in FORPOI$, if A is a theorem of $\mathcal{L}$ then $\models_{\mathcal{L}} A$.*

Proof By induction on the length of the proof of α and on the length of the proof of A.
$\dashv$

5 Completeness

This section presents the proof of the completeness of K with respect to the class of all connected basic models, the proof of the completeness of INC with respect to the class of all models of incidence and the proof of the completeness of PRO with respect to the class of all projective models. These proofs use the techniques of the canonical model.

5.1 Canonical model

Let $\mathcal{L} \in \{K, INC, PRO\}$. The canonical model of $\mathcal{L}$ is the structure of the form $\mathcal{M}_{\mathcal{L}} = (Li_{\mathcal{L}}, Po_{\mathcal{L}}, on_{\mathcal{L}}, in_{\mathcal{L}}, R_{\mathcal{L}}, V_{\mathcal{L}})$ where $Li_{\mathcal{L}}$ is the set of the maximal and $\mathcal{L}$-consistent subsets of $FORLIN$, $Po_{\mathcal{L}}$ is the set of the maximal and $\mathcal{L}$-consistent subsets of $FORPOI$, $on_{\mathcal{L}}$ is the binary relation on $Li_{\mathcal{L}}$ and $Po_{\mathcal{L}}$ defined in the following way :

- For every $x \in Li_{\mathcal{L}}$, $on_{\mathcal{L}}(x) = \{X$: for every $A \in FORPOI$, if $[on]A \in x$ then $A \in X\}$.

$in_{\mathcal{L}}$ is the binary relation on $Po_{\mathcal{L}}$ and $Li_{\mathcal{L}}$ defined in the following way :

- For every $X \in Po_{\mathcal{L}}$, $in_{\mathcal{L}}(X) = \{x$: for every $\alpha \in FORLIN$, if $[in]\alpha \in X$ then $\alpha \in x\}$.

$R_{\mathcal{L}}$ is the mapping of LIN to $\mathcal{P}(Li_{\mathcal{L}})$ defined in the following way :

- For every $\pi \in LIN$, $R_{\mathcal{L}}(\pi) = \{x : \pi \in x\}$.

and $V_{\mathcal{L}}$ is the mapping of POI to $\mathcal{P}(Po_{\mathcal{L}})$ defined in the following way :

- For every $p \in POI$, $V_{\mathcal{L}}(p) = \{X : p \in X\}$.

The reader may easily verify that :

Proposition 11 *The relational structure of the form $(Li_{\mathcal{L}}, Po_{\mathcal{L}}, on_{\mathcal{L}}, in_{\mathcal{L}})$ is a basic frame.*

Moreover :

Proposition 12 *For every $\alpha \in FORLIN$, $\widetilde{R_{\mathcal{L}}}(\alpha) = \{x : \alpha \in x\}$. Moreover, for every $A \in FORPOI$, $\widetilde{V_{\mathcal{L}}}(A) = \{X : A \in X\}$.*

Proof By induction on the complexity of α and on the complexity of A.
$\dashv$

5.2 Connected submodel

Direct calculations would lead to the conclusion that :

Theorem 6 *Every connected submodel of $\mathcal{M}_K$ is a basic model.*

Theorem 7 *Every connected submodel of $\mathcal{M}_{INC}$ is a model of incidence.*

Proof Let $\mathcal{M}^{\circ}_{INC} = (Li, Po, on, in, R, V)$ be a connected submodel of $\mathcal{M}_{INC}$. For every $X, Y \in Po$, let $x_0 = \{\alpha : [in]\alpha \in X\} \cup \{\beta : [in]\beta \in Y\}$. If x_0 is not INC-consistent then there exists $\alpha \in FORLIN$ and there exists $\beta \in FORLIN$ such that $[in]\alpha \in X$, $[in]\beta \in Y$ and $\{\alpha, \beta\}$ is not INC-consistent. Since $\mathcal{M}^{\circ}_{INC}$ is connected, then there exists $k \geq 0$ and there exists $X_0, \ldots, X_k \in Po$ such that :

- $X_0 = X$.
- For every $l \in \{1, \ldots, k\}$, $in(X_{l-1}) \cap in(X_l) \neq \emptyset$.
- $X_k = Y$.

Since $[in]\alpha \in X$, then $\lambda\langle in\rangle\alpha \in X$, for every modality λ of type *point* to *point*. Consequently, $\langle in\rangle\alpha \in Y$ — a contradiction. Consequently, x_0 is INC-consistent and there exists $x \in Li$ such that $x \in in(X)$ and $x \in in(Y)$.
$\dashv$

Theorem 8 *Every connected submodel of $\mathcal{M}_{PRO}$ is a projective model.*

Proof Let $\mathcal{M}^{\circ}_{PRO} = (Li, Po, on, in, R, V)$ be a connected submodel of $\mathcal{M}_{PRO}$. For every $x, y \in Li$, let $X_0 = \{A : [on]A \in x\} \cup \{B : [on]B \in y\}$. If X_0 is not PRO-consistent then there exists $A \in FORPOI$ and there exists $B \in FORPOI$ such that $[on]A \in x$, $[on]B \in y$ and $\{A, B\}$ is not PRO-consistent. Since $\mathcal{M}^{\circ}_{PRO}$ is connected, then there exists $k \geq 0$ and there exists $x_0, \ldots, x_k \in Li$ such that :

- $x_0 = x$.
- For every $l \in \{1, \ldots, k\}$, $on(x_{l-1}) \cap on(x_l) \neq \emptyset$.
- $x_k = y$.

Since $[on]A \in x$, then $\lambda\langle on\rangle A \in x$, for every modality λ of type *line* to *line*. Consequently, $\langle on\rangle A \in y$ — a contradiction. Consequently, X_0 is PRO-consistent and there exists $X \in Po$ such that $X \in on(x)$ and $X \in on(y)$.
$\dashv$

5.3 Completeness

Consequently :

Theorem 9 *For every $\alpha \in FORLIN$, if $\models_{K^\circ} \alpha$ then α is a theorem of K. Moreover, for every $A \in FORPOI$, if $\models_{K^\circ} A$ then A is a theorem of K.*

Proof If α is not a theorem of K then there exists $x \in Li_K$ such that $\alpha \notin x$. According to the proposition 12, $x \notin \widetilde{R_K}(\alpha)$. Consequently, $x \not\models_{\mathcal{M}_K} \alpha$. According to the theorem 1, $x \not\models_{\mathcal{M}^x_K} \alpha$ ($\mathcal{M}^x_K$ being the connected submodel of $\mathcal{M}_K$ containing x). Consequently, $\not\models_{\mathcal{M}^x_K} \alpha$. Consequently, $\not\models_{K^\circ} \alpha$.
$\dashv$

Consequently :

Theorem 10 *For every $\alpha \in FORLIN$, α is a theorem of K iff $\models_K \alpha$ iff $\models_{K^\circ} \alpha$ iff $\models_{K^\circ_f} \alpha$. Moreover, for every $A \in FORPOI$, A is a theorem of K iff $\models_K A$ iff $\models_{K^\circ} A$ iff $\models_{K^\circ_f} A$.*

Moreover :

Theorem 11 *Let $\mathcal{L} \in \{INC, PRO\}$. For every $\alpha \in FORLIN$, if $\models_{\mathcal{L}} \alpha$ then α is a theorem of $\mathcal{L}$. Moreover, for every $A \in FORPOI$, if $\models_{\mathcal{L}} A$ then A is a theorem of $\mathcal{L}$.*

Proof If α is not a theorem of $\mathcal{L}$ then there exists $x \in Li_{\mathcal{L}}$ such that $\alpha \notin x$. According to the proposition 12, $x \notin \widetilde{R_{\mathcal{L}}}(\alpha)$. Consequently, $x \not\models_{\mathcal{M}_{\mathcal{L}}} \alpha$. According to the theorem 1, $x \not\models_{\mathcal{M}_{\mathcal{L}}^x} \alpha$ ($\mathcal{M}_{\mathcal{L}}^x$ being the connected submodel of $\mathcal{M}_{\mathcal{L}}$ containing x). Consequently, $\not\models_{\mathcal{M}_{\mathcal{L}}^x} \alpha$ and $\not\models_{\mathcal{L}} \alpha$.
$\dashv$
Consequently :

Theorem 12 *Let $\mathcal{L} \in \{INC, PRO\}$. For every $\alpha \in FORLIN$, α is a theorem of $\mathcal{L}$ iff $\models_{\mathcal{L}} \alpha$ iff $\models_{\mathcal{L}_f} \alpha$. Moreover, for every $A \in FORPOI$, A is a theorem of $\mathcal{L}$ iff $\models_{\mathcal{L}} A$ iff $\models_{\mathcal{L}_f} A$.*

5.4 The finite model property

Consequently :

Theorem 13 *Let $\mathcal{L} \in \{K, INC, PRO\}$. $\mathcal{L}$ has the finite model property.*

Proof According to the theorems 10 and 12, $\mathcal{L}$ is sound and complete with respect to a class of finite models. Consequently, $\mathcal{L}$ has the finite model property.
$\dashv$
Consequently :

Theorem 14 *Let $\mathcal{L} \in \{K, INC, PRO\}$. $\mathcal{L}$ is decidable.*

Proof $\mathcal{L}$ is finitely axiomatizable and, according to the theorem 13, $\mathcal{L}$ has the finite model property. Consequently, there exists an effective procedure for deciding whether a formula is a theorem of $\mathcal{L}$ or not.
$\dashv$

6 Subordination

This section presents the techniques of the frame of subordination.

6.1 The frame of subordination

The frame of subordination of the modal multilogic of geometry is the structure of the form $\mathcal{F}_\sigma = (Li_\sigma, Po_\sigma, on_\sigma, in_\sigma)$ where Li_σ and Po_σ are the sets defined by induction in the following way :

- $0_{Li} \in Li_\sigma$.
- $0_{Po} \in Po_\sigma$.
- For every $X \in Po_\sigma$ and for every $n \geq 1$, $Xn \in Li_\sigma$.
- For every $x \in Li_\sigma$ and for every $n \geq 1$, $xn \in Po_\sigma$.

on_σ is the binary relation on Li_σ and Po_σ defined in the following way :

- $on_\sigma(0_{Li}) = \{0_{Po}\} \cup \{0_{Li}n : n \geq 1\}$.
- For every $X \in Po_\sigma$ and for every $n \geq 1$, $on_\sigma(Xn) = \{X\} \cup \{Xnn' : n' \geq 1\}$.

and in_σ is the binary relation on Po_σ and Li_σ defined in the following way :

- $in_\sigma(0_{Po}) = \{0_{Li}\} \cup \{0_{Po}n : n \geq 1\}$.
- For every $x \in Li_\sigma$ and for every $n \geq 1$, $in_\sigma(xn) = \{x\} \cup \{xnn' : n' \geq 1\}$.

Our definition yields the following result :

Theorem 15 *$\mathcal{F}_\sigma$ is a countable, normal and connected basic frame.*

6.2 The function of maximality

Let $\mathcal{L} \in \{K, INC, PRO\}$ and $\mathcal{F} = (Li, Po, on, in)$ be a basic frame. An $\mathcal{L}$-function of maximality on $\mathcal{F}$ is a structure of the form (S, W) where S is a mapping of Li to the set of the maximal and $\mathcal{L}$-consistent subsets of $FORLIN$ and W is a mapping of Po to the set of the maximal and $\mathcal{L}$-consistent subsets of $FORPOI$ such that :

- For every $x \in Li$ and for every $A \in FORPOI$, $[on]A \in S(x)$ iff, for every $X \in on(x)$, $A \in W(X)$.
- For every $X \in Po$ and for every $\alpha \in FORLIN$, $[in]\alpha \in W(X)$ iff, for every $x \in in(X)$, $\alpha \in S(x)$.

The mapping R_S of LIN to $\mathcal{P}(Li)$ and the mapping V_W of POI to $\mathcal{P}(Po)$ are defined in the following way :

- For every $\pi \in LIN$, $R_S(\pi) = \{x : \pi \in S(x)\}$.
- For every $p \in POI$, $V_W(p) = \{X : p \in W(X)\}$.

Proposition 13 *For every $\alpha \in FORLIN$, $\widetilde{R_S}(\alpha) = \{x : \alpha \in S(x)\}$. Moreover, for every $A \in FORPOI$, $\widetilde{V_W}(A) = \{X : A \in W(X)\}$.*

Proof By induction on the complexity of α and on the complexity of A.
$\dashv$

6.3 The lemma of subordination

Theorem 16 *Let $\mathcal{L} \in \{K, INC, PRO\}$. For every $\alpha \in FORLIN$, if α is not a theorem of $\mathcal{L}$ then there exists an $\mathcal{L}$-function of maximality (S_σ, W_σ) on $\mathcal{F}_\sigma$ such that $\alpha \notin S_\sigma(0_{Li})$. Moreover, for every $A \in FORPOI$, if A is not a theorem of $\mathcal{L}$ then there exists an $\mathcal{L}$-function of maximality (S_σ, W_σ) on $\mathcal{F}_\sigma$ such that $A \notin W_\sigma(0_{Po})$.*

Proof If α is not a theorem of $\mathcal{L}$ then the mapping S_σ of Li_σ to the set of the maximal and $\mathcal{L}$-consistent subsets of $FORLIN$ and the mapping W_σ of Po_σ to the set of the maximal and $\mathcal{L}$-consistent subsets of $FORPOI$ are defined by induction in the following way :

- Let $S_\sigma(0_{Li})$ be a maximal and $\mathcal{L}$-consistent subset of $FORLIN$ not containing $\{\alpha\}$.
- Let $W_\sigma(0_{Po})$ be a maximal and $\mathcal{L}$-consistent subset of $FORPOI$ containing $\{A : [on]A \in S_\sigma(0_{Li})\}$.
- For every $X \in Po_\sigma$, let $\alpha_1, \alpha_2, \ldots$ be a list of the set $\{\alpha : \langle in\rangle\alpha \in W_\sigma(X)\}$. For every $n \geq 1$, let $S_\sigma(Xn)$ be a maximal and $\mathcal{L}$-consistent subset of $FORLIN$ containing $\{\alpha_n\} \cup \{\beta : [in]\beta \in W_\sigma(X)\}$.
- For every $x \in Li_\sigma$, let $A_1, A_2, \ldots$ be a list of the set $\{A : \langle on\rangle A \in S_\sigma(x)\}$. For every $n \geq 1$, let $W_\sigma(xn)$ be a maximal and $\mathcal{L}$-consistent subset of $FORPOI$ containing $\{A_n\} \cup \{B : [on]B \in S_\sigma(x)\}$.

Direct calculations would lead to the conclusion that (S_σ, W_σ) is an $\mathcal{L}$-function of maximality on $\mathcal{F}_\sigma$.
$\dashv$

7 Normal completeness

This section presents the proof of the completeness of K with respect to the class of all countable, normal and connected basic models, the proof of the completeness of INC with respect to the class of all countable and normal models of incidence and the proof of the completeness of PRO with respect to the class of all countable and normal projective models. These proofs use the techniques of the frame of subordination.

7.1 Normal completeness of K

Theorem 17 *For every $\alpha \in FORLIN$, if $\models_{K^*} \alpha$ then α is a theorem of K. Moreover, for every $A \in FORPOI$, if $\models_{K^*} A$ then A is a theorem of K.*

Proof If α is not a theorem of K then, according to the theorem 16, there exists a K-function of maximality (S_σ, W_σ) on $\mathcal{F}_\sigma$ such that $\alpha \notin S_\sigma(0_{Li})$. Let $\mathcal{M}_{S_\sigma,W_\sigma} = (Li_\sigma, Po_\sigma, on_\sigma, in_\sigma, R_{S_\sigma}, V_{W_\sigma})$ be the basic model on $\mathcal{F}_\sigma$ defined from (S_σ, W_σ). According to the proposition 13, $0_{Li} \notin \widetilde{R_{S_\sigma}}(\alpha)$, $0_{Li} \not\models_{\mathcal{M}_{S_\sigma,W_\sigma}} \alpha$, $\not\models_{\mathcal{M}_{S_\sigma,W_\sigma}} \alpha$ and $\not\models_{K^*} \alpha$.
$\dashv$

Consequently :

Theorem 18 *For every $\alpha \in FORLIN$, α is a theorem of K iff $\models_{K^*}$ α. Moreover, for every $A \in FORPOI$, A is a theorem of K iff $\models_{K^*} A$.*

7.2 Normal completeness of INC

Theorem 19 *For every $\alpha \in FORLIN$, if $\models_{INC^*} \alpha$ then α is a theorem of INC. Moreover, for every $A \in FORPOI$, if $\models_{INC^*} A$ then A is a theorem of INC.*

Proof If α is not a theorem of INC then, according to the theorem 16, there exists an INC-function of maximality (S_σ, W_σ) on $\mathcal{F}_\sigma$ such that $\alpha \notin S_\sigma(0_{Li})$. According to the annex A, $\mathcal{F}_\sigma$ can be gradually extended into a countable and normal frame of incidence $\mathcal{F}^\circ = (Li^\circ, Po^\circ, on^\circ, in^\circ)$ on which there is an INC-function of maximality (S°, W°) extending (S_σ, W_σ). Let $\mathcal{M}^\circ_{S^\circ,W^\circ} = (Li^\circ, Po^\circ, on^\circ, in^\circ, R_{S^\circ}, V_{W^\circ})$ be the basic model on $\mathcal{F}^\circ$ defined from (S°, W°). According to the

proposition 13, $0_{Li} \notin \widetilde{R_{S^\circ}}(\alpha)$, $0_{Li} \not\models_{\mathcal{M}^\circ_{S^\circ,W^\circ}} \alpha$, $\not\models_{\mathcal{M}^\circ_{S^\circ,W^\circ}} \alpha$ and $\not\models_{INC^*} \alpha$.
$\dashv$
Consequently :

Theorem 20 *For every $\alpha \in FORLIN$, α is a theorem of INC iff $\models_{INC^*} \alpha$. Moreover, for every $A \in FORPOI$, A is a theorem of INC iff $\models_{INC^*} A$.*

7.3 Normal completeness of PRO

Theorem 21 *For every $\alpha \in FORLIN$, if $\models_{PRO^*} \alpha$ then α is a theorem of PRO. Moreover, for every $A \in FORPOI$, if $\models_{PRO^*} A$ then A is a theorem of PRO.*

Proof If α is not a theorem of PRO then, according to the theorem 16, there exists a PRO-function of maximality (S_σ, W_σ) on $\mathcal{F}_\sigma$ such that $\alpha \notin S_\sigma(0_{Li})$. According to the annex B, $\mathcal{F}_\sigma$ can be gradually extended into a countable and normal projective frame $\mathcal{F}^\circ = (Li^\circ, Po^\circ, on^\circ, in^\circ)$ on which there is a PRO-function of maximality (S°, W°) extending (S_σ, W_σ). Let $\mathcal{M}^\circ_{S^\circ,W^\circ} = (Li^\circ, Po^\circ, on^\circ, in^\circ, R_{S^\circ}, V_{W^\circ})$ be the basic model on $\mathcal{F}^\circ$ defined from (S°, W°). According to the proposition 13, $0_{Li} \notin \widetilde{R_{S^\circ}}(\alpha)$, $0_{Li} \not\models_{\mathcal{M}^\circ_{S^\circ,W^\circ}} \alpha$, $\not\models_{\mathcal{M}^\circ_{S^\circ,W^\circ}} \alpha$ and $\not\models_{PRO^*} \alpha$.
$\dashv$
Consequently :

Theorem 22 *For every $\alpha \in FORLIN$, α is a theorem of PRO iff $\models_{PRO^*} \alpha$. Moreover, for every $A \in FORPOI$, A is a theorem of PRO iff $\models_{PRO^*} A$.*

8 The extended modal multilogic of geometry

Now, the question is whether the method explained above can be applied as well to other mathematical models of space. Relating to this question, one should in the first place examine closely the potential of a complete axiomatization of the modal multilogic of orthogonal geometry and the potential of a complete axiomatization of the modal multilogic of affine geometry.

8.1 Language

The linguistic basis of the extended modal multilogic of geometry is the modal multilogic of geometry enriched with the modal operator $\Box$ such that :

- For every $\alpha \in FORLIN$, $\Box\alpha \in FORLIN$.

For every $\alpha \in FORLIN$, let $\Diamond\alpha = \neg\Box\neg\alpha$.

- For every $\alpha \in FORLIN$, the complex formula $\Box\alpha$ of type *line* signifies "in every *line orthogonal*, *parallel* with the current *line*, it is the case that α".

The modalities of type *line* to *line* and the modalities of type *point* to *point* are defined by induction in the following way :

- For every $k \geq 0$, $\Box^k$ is a modality of type *line* to *line*.
- The empty modality is a modality of type *point* to *point*.
- For every modality λ of type *line* to *line* and for every $k \geq 0$, $[on][in]\Box^k\lambda$ is a modality of type *line* to *line*.
- For every modality λ of type *point* to *point* and for every $k \geq 0$, $[in]\Box^k[on]\lambda$ is a modality of type *point* to *point*.

8.2 Semantical study

An extended frame is a relational structure of the form $\mathcal{F} = (Li, Po, on, in, \bowtie$ $)$ where $\bowtie$ is a binary relation on Li such that :

- (Li, Po, on, in) is a basic frame.
- For every $x \in Li$, $\bowtie(x) \neq \emptyset$ (every *line* is *orthogonal*, *parallel* with at least one *line*).
- For every $x, y \in Li$, if $y \in \bowtie(x)$ then $x \in \bowtie(y)$ (if a first *line* is *orthogonal*, *parallel* with a second *line* then the second *line* is *orthogonal*, *parallel* with the first *line*).

$\mathcal{F}$ is normal when :

- (Li, Po, on, in) is normal.
- For every $x \in Li$, for every $X \in Po$ and for every $y, z \in Li$, if $\{y, z\} \subseteq \bowtie(x) \cap in(X)$ then $y = z$ (if a *line* and a *point* are *orthogonal*, *parallel* and *incident* with two *lines* then the two *lines* are equal).

$\mathcal{F}$ is connected when :

- For every $X, Y \in Po$, there exists $k \geq 0$ and there exists $X_0, \ldots, X_k \in Po$ such that :
  - $X_0 = X$.
  - For every $l \in \{1, \ldots, k\}$, there exists $m \geq 0$ such that $in(X_{l-1}) \cap \bowtie^m (in(X_l)) \neq \emptyset$.
  - $X_k = Y$.

It is easy to verify that :

Proposition 14 *If $\mathcal{F}$ is connected then :*

- *For every $x \in Li$ and for every $X \in Po$, there exists $k \geq 0$ and there exists $x_0, \ldots, x_k \in Li$ such that :*
  - *$x_0 = x$.*
  - *For every $l \in \{1, \ldots, k\}$, there exists $m \geq 0$ such that $on(x_{l-1}) \cap on(\bowtie^m (x_l)) \neq \emptyset$.*
  - *$x_k \in in(X)$.*

$\mathcal{F}$ is an orthogonal frame :

- For every $x \in Li$ and for every $X \in Po$, $\bowtie (x) \cap in(X) \neq \emptyset$ (a *line* and a *point* are together *orthogonal*, *parallel* and *incident* with at least one *line*).
- For every $X, Y \in Po$, $in(X) \cap in(Y) \neq \emptyset$.
- For every $x, y \in Li$, $\bowtie (x) \cap \bowtie (y) \neq \emptyset$ iff $x = y$ or $on(x) \cap on(y) = \emptyset$.
- For every $x, y, z, t \in Li$, if $y \in \bowtie (x)$, $z \in \bowtie (y)$ and $t \in \bowtie (z)$ then $t \in \bowtie (x)$.

$\mathcal{F}$ is an affine frame when :

- For every $x \in Li$ and for every $X \in Po$, $\bowtie (x) \cap in(X) \neq \emptyset$.
- For every $X, Y \in Po$, $in(X) \cap in(Y) \neq \emptyset$.
- For every $x, y \in Li$, $\bowtie (x) \cap \bowtie (y) \neq \emptyset$ iff $x = y$ or $on(x) \cap on(y) = \emptyset$.
- For every $x, y, z \in Li$, if $y \in \bowtie (x)$ and $z \in \bowtie (y)$ then $z \in \bowtie (x)$.

8.3 Axiomatical study

Together with the axioms and the rules of inference of K, all the instances of the following schema :

- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

and all the instances of the following schemata :

- $\Box\alpha \rightarrow \Diamond\alpha$.
- $\alpha \rightarrow \Box\Diamond\alpha$.

are axioms of E and all the instances of the following schema :

- If α is a theorem then $\Box\alpha$ is a theorem.

are rules of inference of E.

Together with the axioms and the rules of inference of E, all the instances of the following schemata :

- $\Box\alpha \rightarrow \lambda[on]\langle in\rangle\alpha$, for every modality λ of type *line* to *line*.
- $[in]\alpha \rightarrow \lambda\langle in\rangle\alpha$, for every modality λ of type *point* to *point*.
- $\Box\Box\alpha \wedge [on][in]\beta \rightarrow \lambda(\alpha \vee \beta)$, for every modality λ of type *line* to *line*.
- $\Box\alpha \rightarrow \Box\Box\Box\alpha$.

are axioms of ORT, the modal multilogic of orthogonal geometry.

Together with the axioms and the rules of inference of E, all the instances of the following schemata :

- $\Box\alpha \rightarrow \lambda[on]\langle in\rangle\alpha$, for every modality λ of type *line* to *line*.
- $[in]\alpha \rightarrow \lambda\langle in\rangle\alpha$, for every modality λ of type *point* to *point*.
- $\Box\Box\alpha \wedge [on][in]\beta \rightarrow \lambda(\alpha \vee \beta)$, for every modality λ of type *line* to *line*.
- $\Box\alpha \rightarrow \Box\Box\alpha$.

are axioms of AFF, the modal multilogic of affine geometry.

It appears that the method presented above can be applied to the proof of the completeness of ORT and the proof of the completeness of AFF as well :

Theorem 23 *For every $\alpha \in FORLIN$, α is a theorem of ORT iff α is valid in the class of all countable and normal orthogonal models. Moreover, for every $A \in FORPOI$, A is a theorem of ORT iff A is valid in the class of all countable and normal orthogonal models.*

Theorem 24 *For every $\alpha \in FORLIN$, α is a theorem of AFF iff α is valid in the class of all countable and normal affine models. Moreover, for every $A \in FORPOI$, A is a theorem of AFF iff A is valid in the class of all countable and normal affine models.*

9 Conclusion

We have described the frame of incidence as the simplest mathematical model of space : a nonempty set of geometrical beings of type *line*, a nonempty set of geometrical beings of type *point* and a binary relation of *incidence* between *points* and *lines* such that two distinct *points* are together *incident* with exactly one *line*. The projective frame is a frame of incidence in which two distinct *lines* are together *incident* with exactly one *point*. The significance for the geometer of a relational structure such as those ones lies in the fact that there is a very strong connection between projective geometry and the algebraic theory of fields [7].

There is every indication that the proof of the completeness of the modal multilogic of incidence geometry and the proof of the completeness of the modal multilogic of projective geometry are not easy to do. The very important condition of normality (if two *points* are *incident* with two *lines* then either the two *points* are equal or the two *lines* are equal) that we have placed on the frame of incidence and on the projective frame does not correspond to any schema of the language of the modal multilogic of geometry. That is the reason why the proof of the completeness of the modal multilogic of incidence geometry and the proof of the completeness of the modal multilogic of projective geometry use the techniques of the frame of subordination.

We are thinking of studying the following issues :

- The modal multilogic of geometry extended with the modal operator of inequality [23] so that, for every $\alpha \in FORLIN$, $[\neq]\alpha \in FORLIN$ and, for every $A \in FORPOI$, $[\neq]A \in FORPOI$.
- The modal multilogic of projective geometry with the property of Desargues.
- The modal multilogic of orthogonal geometry with the property of the orthocenter.

- The modal multilogic of affine geometry with the property of Desargues.

The reader is kindly invited to examine the possibilities of a complete axiomatization of these modal multilogics. Are these modal multilogics finitely axiomatizable by a structural derivation system ? There is no time to be lost to find a place for space in the modal logic family.

Addendum

During the final step of the preparation of this paper, it has come to the knowledge of the author that Venema [26] had independently achieved some of the above-mentioned results.

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Annex A

Let $\mathcal{F} = (Li, Po, on, in)$ be a countable, normal and connected basic frame and (S, W) be an INC-function of maximality on $\mathcal{F}$. For every $X, Y \in Po$, if $in(X) \cap in(Y) = \emptyset$ then the completion of $\mathcal{F}$ at X and at Y is the structure of the form $\mathcal{F}' = (Li', Po', on', in')$ where :

- $Li' = Li \cup Li_\sigma$.
- $Po' = Po \cup Po_\sigma$.
- $on'_{|Li} = on$.
- $on'(0_{Li}) = on_\sigma(0_{Li}) \cup \{X, Y\}$.
- $on'_{|Li_\sigma \setminus \{0_{Li}\}} = on_\sigma$.
- $in'(X) = in(X) \cup \{0_{Li}\}$.
- $in'(Y) = in(Y) \cup \{0_{Li}\}$.
- $in'_{|Po \setminus \{X,Y\}} = in$.

- $in'_{|Po_\sigma} = in_\sigma$.

Direct calculations would lead to the conclusion that $\mathcal{F}'$ is a countable, normal and connected basic frame. Let $x_0 = \{\alpha : [in]\alpha \in W(X)\} \cup \{\beta : [in]\beta \in W(Y)\}$. If x_0 is not INC-consistent then there exists $\alpha \in FORLIN$ and there exists $\beta \in FORLIN$ such that $[in]\alpha \in W(X)$, $[in]\beta \in W(Y)$ and $\{\alpha, \beta\}$ is not INC-consistent. Since $\mathcal{F}$ is connected, then there exists $k \geq 0$ and there exists $X_0, \ldots, X_k \in Po$ such that :

- $X_0 = X$.
- For every $l \in \{1, \ldots, k\}$, $in(X_{l-1}) \cap in(X_l) \neq \emptyset$.
- $X_k = Y$.

Since $[in]\alpha \in W(X)$, then $\lambda\langle in\rangle\alpha \in W(X)$, for every modality λ of type *point* to *point*. Consequently, $\langle in\rangle\alpha \in W(Y)$ — a contradiction. Consequently, x_0 is INC-consistent. The mapping S' of Li' to the set of the maximal and INC-consistent subsets of $FORLIN$ and the mapping W' of Po' to the set of the maximal and INC-consistent subsets of $FORPOI$ are defined in the following way :

- $S'_{|Li} = S$.
- $W'_{|Po} = W$.
- Let $S'(0_{Li})$ be a maximal and INC-consistent subset of $FORLIN$ containing x_0.
- $S'_{|Li_\sigma \setminus \{0_{Li}\}} = S_\sigma$.
- $W'_{|Po_\sigma} = W_\sigma$.

Direct calculations would lead to the conclusion that (S', W') is an INC-function of maximality on $\mathcal{F}'$. Let $(\mathcal{F}(0), S(0), W(0))$, $(\mathcal{F}(1), S(1), W(1))$, $\ldots$ be the sequence defined by induction in the following way :

- $\mathcal{F}(0) = \mathcal{F}$.
- $S(0) = S$.
- $W(0) = W$.
- For every $k \geq 0$, let $\mathcal{F}(k)$ be a countable, normal and connected basic frame and $(S(k), W(k))$ be an INC-function of maximality on $\mathcal{F}(k)$. Let $X, Y \in Po(k)$ be such that $in(k)(X) \cap in(k)(Y) = \emptyset$. According to the previous line of reasoning, $\mathcal{F}(k)'$ is a countable, normal and connected basic frame and $(S(k)', W(k)')$ is an INC-function of maximality on $\mathcal{F}(k)'$. Let :

- $\mathcal{F}(k+1) = \mathcal{F}(k)'$.
- $S(k+1) = S(k)'$.
- $W(k+1) = W(k)'$.

Let $\mathcal{F}^\circ = (Li^\circ, Po^\circ, on^\circ, in^\circ)$ be the structure defined in the following way :

- $Li^\circ = \bigcup\{Li(k) : k \geq 0\}$.
- $Po^\circ = \bigcup\{Po(k) : k \geq 0\}$.
- on° is the binary relation on Li° and Po° defined in the following way :
  - For every $k \geq 0$ and for every $x \in Li(k)$, $on^\circ(x) = on(k)(x)$.
- in° is the binary relation on Po° and Li° defined in the following way :
  - For every $k \geq 0$ and for every $X \in Po(k)$, $in^\circ(X) = \bigcup\{in(l)(X) : l \geq k\}$.

Direct calculations would lead to the conclusion that $\mathcal{F}^\circ$ is a countable and normal frame of incidence. The mapping S° of Li° to the set of the maximal and INC-consistent subsets of $FORLIN$ and the mapping W° of Po° to the set of the maximal and INC-consistent subsets of $FORPOI$ are defined in the following way :

- For every $k \geq 0$ and for every $x \in Li(k)$, $S^\circ(x) = S(k)(x)$.
- For every $k \geq 0$ and for every $X \in Po(k)$, $W^\circ(X) = W(k)(X)$.

Direct calculations would lead to the conclusion that (S°, W°) is an INC-function of maximality on $\mathcal{F}^\circ$.

Annex B

Let $\mathcal{F} = (Li, Po, on, in)$ be a countable, normal and connected basic frame and (S, W) be a PRO-function of maximality on $\mathcal{F}$. For every $x, y \in Li$, if $on(x) \cap on(y) = \emptyset$ then the completion of $\mathcal{F}$ at x and at y is the structure of the form $\mathcal{F}'' = (Li'', Po'', on'', in'')$ where :

- $Li'' = Li \cup Li_\sigma$.
- $Po'' = Po \cup Po_\sigma$.
- $on''(x) = on(x) \cup \{0_{Po}\}$.

- $on''(y) = on(y) \cup \{0_{Po}\}$.
- $on''_{|Li\setminus\{x,y\}} = on$.
- $on''_{|Li_\sigma} = on_\sigma$.
- $in''_{|Po} = in$.
- $in''(0_{Po}) = in_\sigma(0_{Po}) \cup \{x, y\}$.
- $in''_{|Po_\sigma\setminus\{0_{Po}\}} = in_\sigma$.

Direct calculations would lead to the conclusion that $\mathcal{F}''$ is a countable, normal and connected basic frame. Let $X_0 = \{A : [on]A \in S(x)\} \cup \{B : [on]B \in S(y)\}$. If X_0 is not PRO-consistent then there exists $A \in FORPOI$ and there exists $B \in FORPOI$ such that $[on]A \in S(x)$, $[on]B \in S(y)$ and $\{A, B\}$ is not PRO-consistent. Since $\mathcal{F}$ is connected, then there exists $k \geq 0$ and there exists $x_0, \ldots, x_k \in Li$ such that :

- $x_0 = x$.
- For every $l \in \{1, \ldots, k\}$, $on(x_{l-1}) \cap on(x_l) \neq \emptyset$.
- $x_k = y$.

Since $[on]A \in S(x)$, then $\lambda\langle on\rangle A \in S(x)$, for every modality λ of type *line* to *line*. Consequently, $\langle on\rangle A \in S(y)$ — a contradiction. Consequently, X_0 is PRO-consistent. The mapping S'' of Li'' to the set of the maximal and PRO-consistent subsets of $FORLIN$ and the mapping W'' of Po'' to the set of the maximal and PRO-consistent subsets of $FORPOI$ are defined in the following way :

- $S''_{|Li} = S$.
- $W''_{|Po} = W$.
- $S''_{|Li_\sigma} = S_\sigma$.
- Let $W''(0_{Po})$ be a maximal and PRO-consistent subset of $FORPOI$ containing X_0.
- $W''_{|Po_\sigma\setminus\{0_{Po}\}} = W_\sigma$.

Direct calculations would lead to the conclusion that (S'', W'') is a PRO-function of maximality on $\mathcal{F}''$. Let $(\mathcal{F}(0), S(0), W(0))$, $(\mathcal{F}(1), S(1), W(1))$, $\ldots$ be the sequence defined by induction in the following way :

- $\mathcal{F}(0) = \mathcal{F}$.
- $S(0) = S$.

- $W(0) = W$.
- For every $k \geq 0$, let $\mathcal{F}(2 \times k)$ be a countable, normal and connected basic frame and $(S(2 \times k), W(2 \times k))$ be a *PRO*-function of maximality on $\mathcal{F}(2 \times k)$. Let $x, y \in Li(2 \times k)$ be such that $on(2 \times k)(x) \cap on(2 \times k)(y) = \emptyset$. According to the previous line of reasoning, $\mathcal{F}(2 \times k)''$ is a countable, normal and connected basic frame and $(S(2 \times k)'', W(2 \times k)'')$ is a *PRO*-function of maximality on $\mathcal{F}(2 \times k)''$. Let :
  - $\mathcal{F}(2 \times k + 1) = \mathcal{F}(2 \times k)''$.
  - $S(2 \times k + 1) = S(2 \times k)''$.
  - $W(2 \times k + 1) = W(2 \times k)''$.
- For every $k \geq 0$, let $\mathcal{F}(2 \times k + 1)$ be a countable, normal and connected basic frame and $(S(2 \times k+1), W(2 \times k+1))$ be a *PRO*-function of maximality on $\mathcal{F}(2 \times k + 1)$. Let $X, Y \in Po(2 \times k + 1)$ be such that $in(2 \times k + 1)(X) \cap in(2 \times k + 1)(Y) = \emptyset$. According to the line of reasoning of the previous annex, $\mathcal{F}(2 \times k + 1)'$ is a countable, normal and connected basic frame and $(S(2 \times k+1)', W(2 \times k+1)')$ is a *PRO*-function of maximality on $\mathcal{F}(2 \times k + 1)'$. Let :
  - $\mathcal{F}(2 \times k + 2) = \mathcal{F}(2 \times k + 1)'$.
  - $S(2 \times k + 2) = S(2 \times k + 1)'$.
  - $W(2 \times k + 2) = W(2 \times k + 1)'$.

Let $\mathcal{F}^\circ = (Li^\circ, Po^\circ, on^\circ, in^\circ)$ be the structure defined in the following way :

- $Li^\circ = \bigcup\{Li(k) : k \geq 0\}$.
- $Po^\circ = \bigcup\{Po(k) : k \geq 0\}$.
- on° is the binary relation on Li° and Po° defined in the following way :
  - For every $k \geq 0$ and for every $x \in Li(k)$, $on^\circ(x) = \bigcup\{on(l)(x) : l \geq k\}$.
- in° is the binary relation on Po° and Li° defined in the following way :
  - For every $k \geq 0$ and for every $X \in Po(k)$, $in^\circ(X) = \bigcup\{in(l)(X) : l \geq k\}$.

Direct calculations would lead to the conclusion that $\mathcal{F}^\circ$ is a countable and normal projective frame. The mapping S° of Li° to the set of the maximal and PRO-consistent subsets of $FORLIN$ and the mapping W° of Po° to the set of the maximal and PRO-consistent subsets of $FORPOI$ are defined in the following way :

- For every $k \geq 0$ and for every $x \in Li(k)$, $S^\circ(x) = S(k)(x)$.
- For every $k \geq 0$ and for every $X \in Po(k)$, $W^\circ(X) = W(k)(X)$.

Direct calculations would lead to the conclusion that (S°, W°) is a PRO-function of maximality on $\mathcal{F}^\circ$.